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Feynman-Kac Theorem

$$\begin{cases} \partial_t g + r s \partial_s g + \frac{1}{2}\sigma^2 s^2 \partial_{ss} g = r g \\ \qquad\qquad\qquad g(T,s) = \varphi(s) \end{cases}$$

$$g(t,s) \overset{?}{=} e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}[\varphi(S_T) \mid S_t = s]$$

$$S_T \overset{d}{=} S\, e^{(r-\frac{1}{2}\sigma^2)(T-t) + \sigma(\underbrace{\tilde{W}_T - \tilde{W}_t}_{\mathbb{Q}\text{-B.motion}})}$$

NB: $h(t,s) \triangleq \mathbb{E}^{\mathbb{Q}}[\varphi(S_T) \mid S_t = s]$

$h_t = h(t, S_t)$ is a $\mathbb{Q}$-martingale

(i.e. $\mathbb{E}^{\mathbb{Q}}[h_u \mid h_t] = h_t$, $T \geq u > t$)

today's value is the best estimate of the future expectation.

let's check ...

$\mathbb{E}^{\mathbb{Q}}[h_u \mid h_t]$

$$= \mathbb{E}^{\mathbb{Q}}[\underbrace{\mathbb{E}^{\mathbb{Q}}[\varphi(S_T) \mid S_u]}_{h_u} \mid h_t] \qquad S_t$$

$$= \mathbb{E}^{\mathbb{Q}}[\varphi(S_T) \mid S_t] = h_t$$

($\mathbb{E}[\mathbb{E}[X \mid Y] \mid Z] = \mathbb{E}[X \mid Z]$, $Y > Z$)

so: $\mathbb{E}^{\mathbb{Q}}[h_u \mid h_t] = h_t$

$\Rightarrow \mathbb{E}^{\mathbb{Q}}[h_u \mid h_t] - h_t = 0$

$\Rightarrow \mathbb{E}^{\mathbb{Q}}[(h_u - h_t) \mid h_t] = 0$ (since h_t is a "constant" under the $\mathbb{E}[\cdot \mid h_t]$)

$\hookrightarrow u = t + \varepsilon$

$$h_{t+\varepsilon} - h_t = \int_t^{t+\varepsilon} \partial_s h_u \, dS_u + \int_t^{t+\varepsilon} \partial_t h_u \, du + \frac{1}{2}\sigma^2 \int_t^{t+\varepsilon} S_u^2 \, \partial_{ss} h_u \, du$$

$$\left(dh = \partial_s h \, dS_t + \partial_t h \, dt + \tfrac{1}{2} \sigma^2 S_t^2 \partial_{ss} h \, dt \right)$$

recall our guess was $S_T \overset{d}{=} S_t \, e^{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)}$

Further guess: $S_t = S_0 \, e^{(r-\frac{1}{2}\sigma^2)t + \sigma W_t}$

$$\Leftrightarrow \frac{dS_t}{S_t} = r \, dt + \sigma \, dW_t$$

then $\mathbb{E}^{\mathbb{Q}}[h_{t+\varepsilon} - h_t \mid h_t] = 0$

$$\Rightarrow \mathbb{E}^{\mathbb{Q}}\left[\int_t^{t+\varepsilon} \left(\partial_s h_u \; r \, S_u + \partial_t h_u + \tfrac{1}{2} \sigma^2 S_u^2 \, \partial_{ss} h_u \right) du \,\Big|\, h_t \right] = 0$$

b/c $\mathbb{E}^{\mathbb{Q}}\left[\int_t^{t+\varepsilon} l_u \, dW_u \right] = 0$

$$\left(\begin{array}{l} \lim\limits_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_a^{a+\varepsilon} l_u \, du = l_a \\ \quad \frac{1}{\varepsilon} \quad l_a \cdot \varepsilon \end{array} \right)$$

$$\Rightarrow \mathbb{E}^{\mathbb{Q}}[\underbrace{r S_t \partial_s h_t + \partial_t h_t + \tfrac{1}{2} \sigma^2 S_t^2 \partial_{ss} h_t}_{\text{is not random } \mathbb{E}^{\mathbb{Q}}[(\cdot) | h_t] = (\cdot)} \mid h_t] = 0$$

$$\Rightarrow \boxed{\partial_t h(t,S) + r S \, \partial_s h(t,S) + \tfrac{1}{2} \sigma^2 S^2 \partial_{ss} h(t,S) = 0}$$

recall that $g(t,S) = e^{-r(T-t)} h(t,S)$

$$\Rightarrow h(t,S) = e^{r(T-t)} g(t,S)$$

$e^{r(T-t)} [$

$$\Rightarrow \quad e \left[-rg + \partial_t g + rS\,\partial_S g + \tfrac{1}{2}\sigma^2 S^2 \partial_{SS} g \right] = 0$$

$$\Rightarrow \quad \boxed{\begin{aligned} &\partial_t g + rS\,\partial_S g + \tfrac{1}{2}\sigma^2 S^2 \partial_{SS} g = rg \\ &g(T,S) = h(T,S) = \Phi(S) \end{aligned}}$$

is B-S PDE !

Feynman-Kac Theorem

If $g(t,S)$ satisfies the PDE

$$\begin{cases} \partial_t g + a(t,S)\,S\,\partial_S g + \frac{1}{2} b^2(t,S)\,S^2\,\partial_{SS} g = c(t,S)\,g \\ g(T,S) = \Phi(S) \end{cases}$$

then g admits a unique sol given by:

$$g(t,S) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T c(u,S_u)\,du} \cdot \Phi(S_T) \;\middle|\; S_t \right]$$

where,

$$dS_t = S_t\,a(t,S_t)\,dt + S_t\,b(t,S_t)\,dW_t^{\mathbb{Q}}$$

$\mathbb{Q}$- B. motn.

$$\begin{cases} \partial_t g + a\,\partial_x g + \frac{1}{2} b^2 \partial_{xx} g = c\,g \\ \qquad g(T,x) = x \quad (x^2) \quad (e^{-x}) \end{cases}$$

solve it ... Feynman-Kac

$$g(t,x) = \mathbb{E}^{Q}\left[e^{-c(T-t)} \cdot X_T \mid X_t = x \right]$$

$$dX_t = a\,dt + b\,dW_t^{Q}$$

$$\Rightarrow X_T = X_t + a(T-t) + b(W_T - W_t)$$

$$\Rightarrow g(t,x) = \mathbb{E}^{Q}\left[e^{-c(T-t)} \left(x + a(T-t) + b(W_T - W_t) \right) \right]$$

$$= e^{-c(T-t)} \left(x + a(T-t) \right)$$

if $g(T,x) = x^2$

$$g(t,x) = e^{-c(T-t)} \mathbb{E}^{Q}\left[\left(x + a(T-t) + b(W_T - W_t) \right)^2 \right]$$

$$= e^{-c(T-t)} \mathbb{E}^{Q}\left[x^2 + a^2(T-t)^2 + b^2 (W_T - W_t)^2 + 2ax(T-t) + 2(x + a(T-t))\,b(W_T - W_t) \right]$$

$$= e^{-c(T-t)} \left(x^2 + a^2 (T-t)^2 + b^2 (T-t) + 2ax(T-t) \right)$$

if $g(T,x) = e^{-x}$

$$g(t,x) = e^{-c(T-t)} \mathbb{E}^{Q}\left[e^{-(x + a(T-t) + b(W_T - W_t))} \right]$$

$$= e^{-c(T-t) - (x + a(T-t))} \cdot \mathbb{E}^{Q}\left[e^{b(W_T - W_t)} \right]$$

$$\hookrightarrow e^{\frac{1}{2} b^2 (T-t)}$$

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Dynamic - Hedging in Incomplete Markets

$$dr_t = \left(\kappa(\theta - r_t)\right)dt + \sigma\, dW_t$$

you cannot trade r to price g !

α_t units of a claim h_t

β_t " " M.M.

-1 " " g_t

market price of risk
(Sharpe ratio)

$$\partial_t g + \underbrace{\left[\left(\kappa(\theta - r_t)\right) - \lambda_t \sigma\right]}\, \partial_r g + \tfrac{1}{2}\sigma^2 \partial_{rr} g = r g$$

in B-S was r itself

typically $\lambda_t = a + b\, r_t$

$\bar{\kappa}(\bar{\theta} - r_t)$

$$\mathbb{P} \longrightarrow \mathbb{Q}$$

κ, θ $\bar{\kappa}, \bar{\theta}$ modify level & rate of mean-reversion.

Feynman-Kac $\Rightarrow$

$$g(t,r) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_u\, du} \cdot \varphi(r_T) \mid r_t = r\right]$$

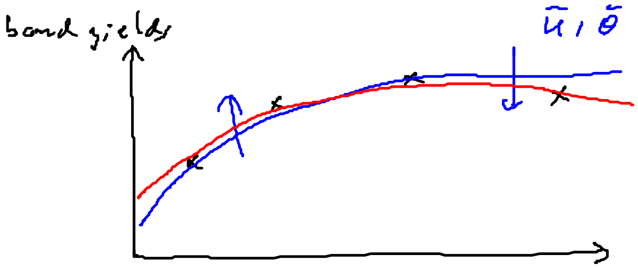
$$dr_t = \bar{\kappa}(\bar{\theta} - r_t)\, dt + \sigma\, dW_t^{\mathbb{Q}}$$

bond price: $\varphi(r_T) = 1$

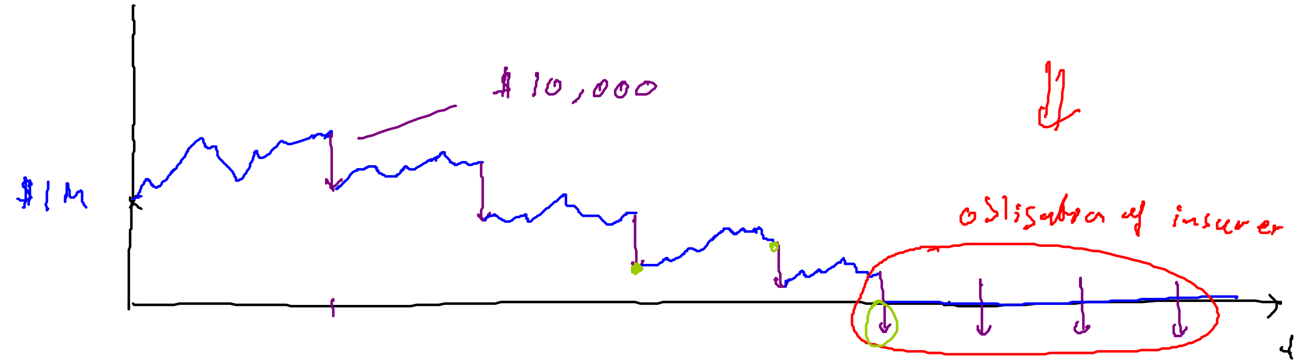
$$g(t,r) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_u\, du} \mid r_t = r\right]$$

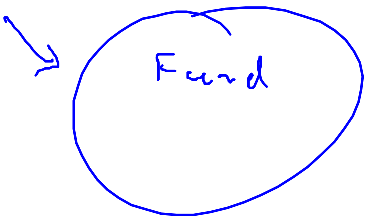
$$\int_t^T r_u\, du \underset{\mathbb{Q}}{\sim} N(m; v)$$

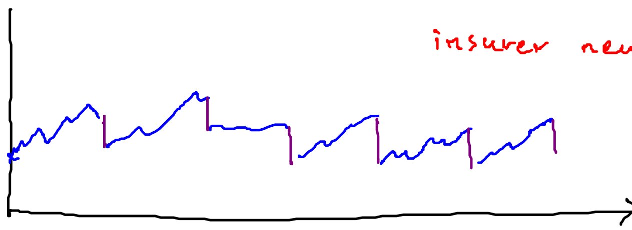
$$g(t,r) = \exp\left\{ -m + \tfrac{1}{2} v \right\}$$



Guaranteed Minimal Withdrawal Benefits







$$F_{T_n^-} = F_{T_{n-1}} \, e^{(r - \frac{1}{2}\sigma^2)\Delta T + \sigma(W_{T_n^-} - W_{T_{n-1}})}$$

$$F_{T_n^+} = (F_{T_n^-} - K)_+$$

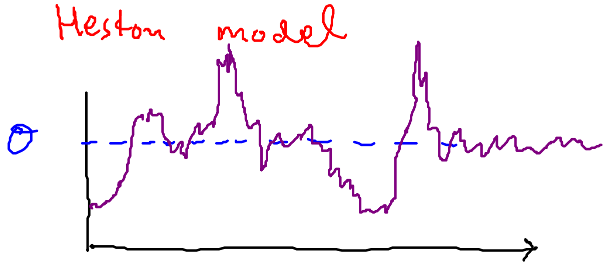
$$\frac{dS_t}{S_t} = r\,dt + \sqrt{v_t}\,dW_t$$

stochastic vol

$$dv_t = \kappa(\theta - v_t)\,dt + \eta\sqrt{v_t}\,dB_t$$

to prevent −ve of v_t.

Heston model



Regime Switching Models:

$\sigma_t = \sigma(H_t)$

Markov chain $(1, 2, \ldots, N)$

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