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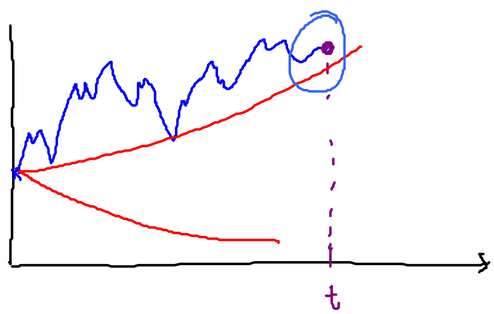
Black-Scholes Partial Differential Equation

value a contingent claim in continuous time.

- $\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t$ ↳ $\mathbb{P}$ - B mtn
- $\frac{dM_t}{M_t} = r\, dt$

→ $S_t = S_0\, e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$

→ $S_t \overset{d}{=} S_0\, e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma \sqrt{t}\, Z}$, $Z \underset{\mathbb{P}}{\sim} N(0,1)$



$S_0 \langle\, S_u ,\ S_d$ $\qquad 1 \langle\, e^{r\Delta t},\ e^{r\Delta t}$ $\qquad C_0 \langle$

$\alpha \qquad \beta$

$C_u = \varphi(S_u) = \alpha S_u + \beta e^{r\Delta t}$

$C_d = \varphi(S_d) = \alpha S_d + \beta e^{r\Delta t}$

↳ by no arb

$C_0 = \alpha S_0 + \beta$
$= e^{-r\Delta t}\, \mathbb{E}^{\mathbb{Q}}[\, C_1 \,]$

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i) set up a strategy α_t - units of S_t
β_t - " - M_t
-1 - " " claim (g_t)

$$V_t = \alpha_t S_t + \beta_t M_t - g_t \leftarrow$$

a) set up to initially cost 0.

$$\Rightarrow V_0 = \alpha_0 S_0 + \beta_0 M_0 - g_0 = 0$$

ii) dynamics of V_t :

$$dV_t = d(\alpha_t S_t) + d(\beta_t M_t) - dg_t$$

$$= d\alpha_t \, S_t + \underline{\alpha_t dS_t} + d[\alpha, S]_t$$

$$+ d\beta_t \, M_t + \underline{\beta_t \, dM_t} + d[\beta, M]_t$$

$$- \underline{dg_t}$$

$$\left(\text{reminder: } [X, Y]_t = \lim_{\|\pi\| \downarrow 0} \sum_k \Delta X_k \, \Delta Y_k \right)$$

$$dV_t = \alpha_t \, dS_t + \beta_t \, dM_t - dg_t$$

self-financing constraint

assume $g_t = g(t, S_t) \in C^{1,2}$

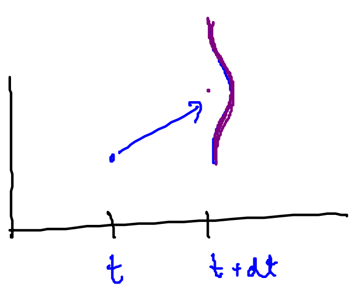
$$\Rightarrow dV_t = \alpha_t (S_t \, \mu \, dt + S_t \, \sigma \, dW_t)$$

$$+ \beta_t \, M_t \, r \, dt$$

$$- \Big[(\partial_t g + \mu S_t \partial_s g + \tfrac{1}{2} \sigma^2 S_t^2 \partial_{ss} g) \, dt$$

$$+ \sigma S_t \partial_s g \, dW_t \Big]$$

$\hookrightarrow$ Itô's lemma on $g(t, S_t)$



* make "instantaneous volatility" equal 0

(i.e. remove local risk)

$\Rightarrow$ choose: $\boxed{\alpha_t = \partial_s g(t, S_t)}$

$$dV_t = \Big(\mu S_t \alpha_t + r M_t \beta_t$$
$$- \big(\partial_t g + \mu S_t \partial_s g + \tfrac{1}{2}\sigma^2 S_t^2 \partial_{ss} g \big) \Big) dt$$
$$+ 0 \cdot dW_t$$

$$= \underbrace{\left(r M_t \beta_t - \partial_t g - \tfrac{1}{2}\sigma^2 S_t^2 \partial_{ss} g \right)}_{\text{guaranteed drift}} dt + 0 \cdot dW_t$$

since $V_0 = 0$ & $dV_t = (\cdot)\, dt + 0 \cdot dW_t$, then to avoid arb we must have $\boxed{(\cdot) = 0}$

so we have $V_0 = 0$ & $dV_t = 0 \Rightarrow V_t = 0$

$$\therefore \quad V_t = \alpha_t S_t + \beta_t M_t - g_t = 0$$

$$\Rightarrow \quad \beta_t M_t = g_t - S_t \partial_s g$$

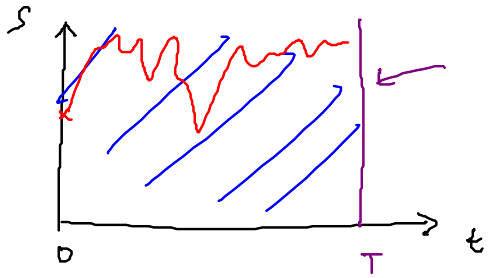
$$\Rightarrow \quad r(g_t - S_t \partial_s g) - \partial_t g - \tfrac{1}{2}\sigma^2 S_t^2 \partial_{ss} g = 0$$

$$\Rightarrow \quad \partial_t g + r S_t \partial_s g + \tfrac{1}{2}\sigma^2 S_t^2 \partial_{ss} g = r g_t$$

$$\forall (t, S_t)$$

$$\partial_t g(t,S) + r S \partial_S g(t,S) + \frac{1}{2}\sigma^2 S^2 \partial_{SS} g(t,S) = r g(t,S)$$

has to hold on $([0,T] \times \mathbb{R}_+)$



$g(T,S) = Q(S)$ payoff

Black-Scholes PDE

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t$$

dynamic hedging argument

recall CRR model

$$S \begin{cases} S e^{\sigma\sqrt{\Delta t}} & p \\ S e^{-\sigma\sqrt{\Delta t}} \end{cases}$$

$$p = \frac{1}{2}\left(1 + \frac{\mu - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right) + o(\Delta t)$$

but $$q = \frac{1}{2}\left(1 + \frac{r - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right) + o(\Delta t)$$

note: if $\varphi(S) = S$ this claim is the asset itself, and its value must be S_t i.e.

$$g(t,S) = S$$

check if it is a solution of Black-Scholes PDE:

$$\partial_t g = 0, \quad \partial_S g = 1, \quad \partial_{SS} g = 0$$

$$\underset{\substack{\shortparallel \\ 0}}{\partial_t g} + r S \underset{\substack{\shortparallel \\ 1}}{\partial_S g} + \tfrac{1}{2}\sigma^2 S^2 \underset{\substack{\shortparallel \\ 0}}{\partial_{SS} g} \overset{?}{=} \underset{\substack{\shortparallel \\ S}}{r g}$$

✓

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e.g. $\varphi(S) = \mathbb{I}_{S>K}$.

expect: $g(t,S) \overset{?}{=} e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}[\mathbb{I}_{S_T>K} \mid S_t = S]$

$$S_T \overset{d}{=} S\, e^{(r-\frac{1}{2}\sigma^2)(T-t) + \sigma\sqrt{T-t}\, Z}$$

$$Z \underset{\mathbb{Q}}{\sim} N(0,1)$$

$$g(t,S) \overset{?}{=} e^{-r(T-t)} \mathbb{Q}(S_T > K)$$

$$= e^{-r(T-t)} \mathbb{Q}\left(Z > \frac{\ln(K/S) - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \right)$$

$$= e^{-r(T-t)} \Phi(d_-)$$

$$d_- = \frac{\ln(S/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$\partial_t g = r\, g + e^{-r(T-t)}\ \Phi'(d_-) \cdot \left[\frac{\ln(S/K)}{\sigma} \frac{1}{2} \frac{1}{(T-t)^{3/2}} + \left(\frac{r-\frac{1}{2}\sigma^2}{\sigma}\right)\cdot\left(-\frac{1}{(T-t)^{1/2}}\right)\right]$$

$$\partial_S g = e^{-r(T-t)}\ \Phi'(d_-)\left[\frac{1}{S\sigma\sqrt{T-t}}\right] \qquad \times\ S r$$

$$\partial_{SS} g = e^{-r(T-t)}\left(\Phi''(d_-)\left[\frac{1}{S\sigma\sqrt{T-t}}\right]^2 + \Phi'(d_-)\left(-\frac{1}{S^2}\ \frac{1}{\sigma\sqrt{T-t}}\right)\right] \qquad \times\ \frac{1}{2}\sigma^2 S^2$$

$$\left(\quad \Phi(x) = \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} \frac{dy}{\sqrt{2\pi}} \right.$$

$$\Phi'(x) = \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}$$

$$\left. \Phi''(x) = -x\ \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} = -x\ \Phi'(x) \right)$$

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e.g. $\quad \varphi(S) = S^{1/2}$

$$g(t,S) \stackrel{?}{=} e^{-r(T-t)}\ \mathbb{E}^{\mathbb{Q}}\left[S_T^{1/2} \mid S_t = S\right]$$

$$S_T \stackrel{d}{=} S\ e^{(r-\frac{1}{2}\sigma^2)(T-t) + \sigma\sqrt{T-t}\,Z}\ , \quad Z \underset{\mathbb{Q}}{\sim} N(0,1)$$

$$\Rightarrow\ g(t,S) \stackrel{?}{=} S^{1/2}\, e^{-\frac{r}{2}(T-t) - \frac{1}{4}\sigma^2(T-t)}\ .\ e^{\frac{1}{2}\left(\frac{1}{2}\sigma\sqrt{T-t}\right)^2}$$

$$= S^{1/2}\ .\ e^{-\frac{r}{2}(T-t) - \frac{\sigma^2}{8}(T-t)}$$

$$= S^{1/2}\ e^{-\left(\frac{r}{2} + \frac{\sigma^2}{8}\right)(T-t)}$$

$$\partial_t g = +\left(\frac{r}{2} + \frac{\sigma^2}{8}\right) g$$

$$\partial_S g = \frac{1}{2} S^{-1/2} e^{-\left(\frac{r}{2} + \frac{\sigma^2}{8}\right)(T-t)} \qquad \times\, r S$$

$$\partial_{SS} g = \frac{1}{2}\left(-\frac{1}{2}\right) S^{-3/2} e^{-\left(\frac{r}{2} + \frac{\sigma^2}{8}\right)(T-t)} \qquad \times\, \frac{1}{2}\sigma^2 S^2$$

$$g\left(\frac{r}{2} + \frac{\sigma^2}{8}\right) + \frac{r}{2} g - \frac{1}{8}\sigma^2 g$$

$$= r g \quad \checkmark$$

b. c. $\quad g(t,S) \xrightarrow[t \uparrow T]{} S^{1/2} = Q(S)$

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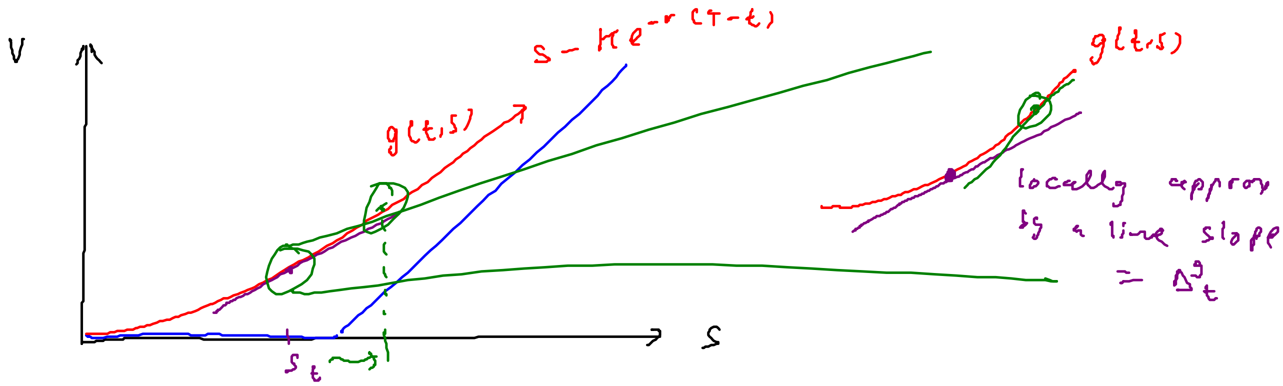
Dynamic Hedging in Discrete Time.



$\Delta^g_t = \partial_S g(t, S_t)$ Delta

suppose we sold a call option & try to hedge it

<u>$t=0$</u> :
- get g_0 by selling claim
- purchase Δ^g_0 of S ... cost $\Delta^g_0 S_0$

$$M_0 = g_0 - \Delta^g_0 S_0$$

<u>$t=\Delta t$</u>
- $M_0 \longrightarrow M_0 e^{r\Delta t}$
- $\Delta^g_0 \longrightarrow \Delta^g_0$

$\Big\}$ $M_0 e^{r\Delta t} + \Delta^g_0 S_{\Delta t}$

rebalance $\Delta^g_0 \to \Delta^g_{\Delta t}$ buy $(\Delta^g_{\Delta t} - \Delta^g_0)$ more units of S

Costs : $(\Delta^g_{\Delta t} - \Delta^g_0) S_{\Delta t}$

$$M_{\Delta t} = M_0 e^{r\Delta t} - (\Delta^g_{\Delta t} - \Delta^g_0) S_{\Delta t}$$

<u>$t = 2\Delta t$</u>:

rebalance $\Delta^g_{\Delta t} \longrightarrow \Delta^g_{2\Delta t}$

$$\Rightarrow \quad M_{2\Delta t} = M_{\Delta t} e^{r\Delta t} - (\Delta^g_{2\Delta t} - \Delta^g_{\Delta t}) S_{2\Delta t}$$

in general after rebalancing

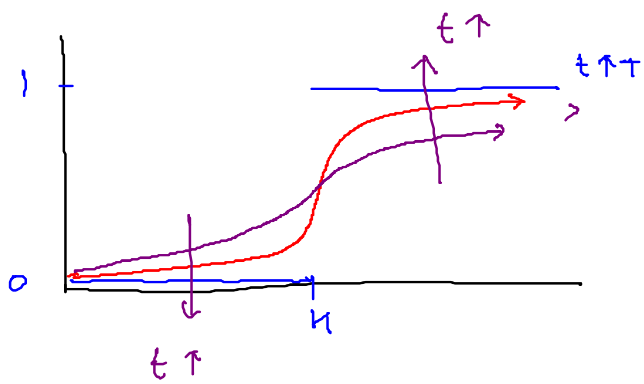
$$\boxed{\begin{aligned} M_{n\Delta t} &= M_{(n-1)\Delta t}\, e^{r\Delta t} - (\Delta^g_{n\Delta t} - \Delta^g_{(n-1)\Delta t})\, S_{n\Delta t} \\ &\qquad - |\Delta^g_{n\Delta t} - \Delta^g_{(n-1)\Delta t}|\, S_{n\Delta t}\,\delta \\ &\Delta^g_{n\Delta t} \text{ of asset} \end{aligned}}$$

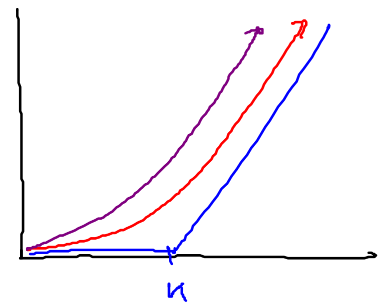
$$P\&L = M_{N\Delta t} + \Delta^g_{N\Delta t}\, S_{N\Delta t} - \varphi(S_{N\Delta t})$$

$$g^{call}(t,S) = S\,\Phi(d_+) - K e^{-r(T-t)}\,\Phi(d_-)$$

$$d_\pm = \frac{\ln(S/K) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

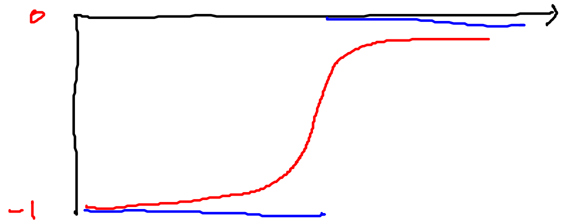
$$\Delta^{call}_t = \Phi(d_+)$$



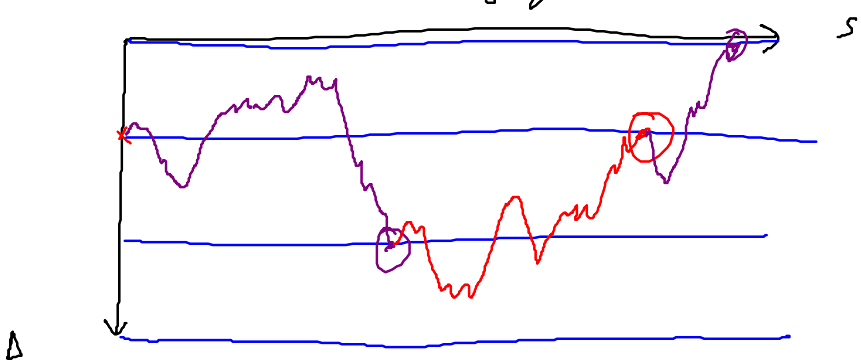


$$C - P = S - K e^{-r(T-t)}$$

$$\Delta^C - \Delta^P = 1 \Rightarrow \Delta^P = \Delta^C - 1$$



Move-based hedging:



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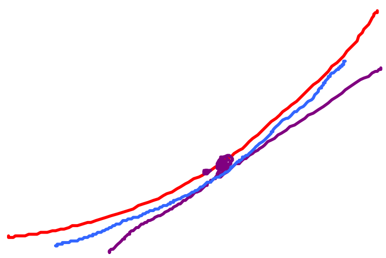
Delta-Gamma Hedging:



↳ quadratic approximation.

$$\Delta g \sim \Delta^g (\Delta S) + \frac{1}{2} (\partial_{SS} g) (\Delta S)^2 + \dots$$

↳ Γ gamma.

$\shortparallel$

$\partial_S \Delta^g$

α_t – units of S_t
β_t – units of M_t
γ_t – units of a second option h_t

$$\Delta^S = \partial_S (S) = 1 \Rightarrow \Gamma^S = \partial_S (1) = 0.$$

$$V = \alpha S + \beta e^{rt} + \gamma h$$

want Δ & Γ of V to match g.

Δ: $\alpha_t + 0 + \gamma_t \Delta^h_t = \Delta^g_t$

Γ: $0 + 0 + \gamma_t \Gamma_t^h = \Gamma_t^g$

$$\Rightarrow \boxed{\begin{aligned} \gamma_t &= \frac{\Gamma_t^g}{\Gamma_t^h} \\ \alpha_t &= \Delta_t^g - \frac{\Gamma_t^g}{\Gamma_t^h} \Delta_t^h \end{aligned}}$$

$$\boxed{\begin{aligned} M_{n\Delta t} = {} & M_{(n-1)\Delta t} \, e^{r\Delta t} \\ & - (\alpha_{n\Delta t} - \alpha_{(n-1)\Delta t}) \, S_{n\Delta t} \\ & - (\gamma_{n\Delta t} - \gamma_{(n-1)\Delta t}) \, H_{n\Delta t} \end{aligned}}$$