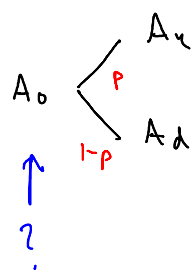


$$A_0 \overset{?}{=} \frac{1}{1+r} \mathbb{E}[A_1]$$

$$= \frac{1}{1+r}\left(p A_u + (1-p) A_d\right)$$

but agents are risk-averse!

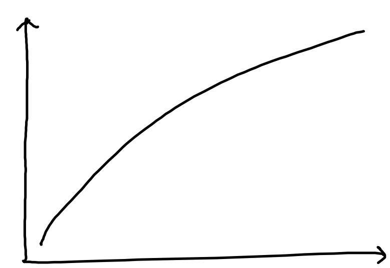
$\mathbb{E}[u(X)]$ – expected utility of terminal wealth

$$X_1 \preceq X_2 \iff \mathbb{E}[u(X_1)] \le \mathbb{E}[u(X_2)]$$

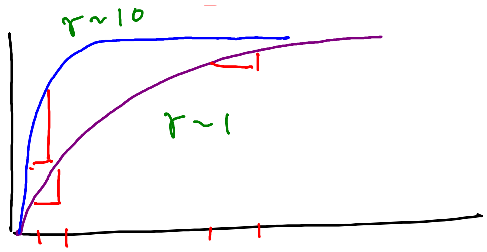
" prefer X_2 to X_1 iff expected utility of X_2 is greater than exp. utl. of X_1 "

$u : \mathbb{R} \mapsto \mathbb{R}$

- $u(x)$ is increasing
- $u(x)$ is concave



$$u(x) = \frac{1 - e^{-\gamma x}}{\gamma}, \quad \gamma > 0 \quad \text{risk-aversion level}$$



$$u(x) \xrightarrow[\gamma \downarrow 0]{} x$$

Indifference Pricing:

find A_0 s.t. $\mathbb{E}[u(X_1)] = \mathbb{E}[u(X_2)]$

"do nothing" "buy asset"

$X_1 = x(1+r)$

$X_2 = (x - A_0)(1+r) + A_1$

asset price at time 1

A_u prob p

A_d prob $1-p$

$$\frac{1 - e^{-\gamma x(1+r)}}{\gamma} = \frac{1 - \mathbb{E}[e^{-\gamma(x-A_0)(1+r) - \gamma A_1}]}{\gamma}$$

$$\Rightarrow \quad e^{-\gamma x(1+r)} = e^{-\gamma(x-A_0)(1+r)} \mathbb{E}[e^{-\gamma A_1}]$$

$$\Rightarrow \quad \boxed{A_0 = -\frac{1}{(1+r)\gamma} \ln \mathbb{E}[e^{-\gamma A_1}]}$$

$e^{\alpha} = 1 + \alpha + o(\alpha)$

risk-neutral agent:

$$\xrightarrow[\gamma \downarrow 0]{} -\frac{1}{(1+r)\gamma} \ln \mathbb{E}[1 - \gamma A_1 + o(\gamma)]$$

$$= -\frac{1}{(1+r)\gamma}\ln\left(1-\gamma\,\mathbb{E}[A_1]+o(\gamma)\right)$$

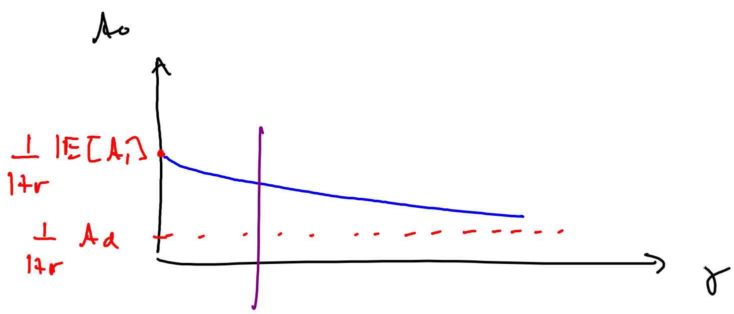
$\ln(1+\alpha)=\alpha+o(\alpha)$

$$= \frac{1}{1+r}\mathbb{E}[A_1]+o(\gamma)$$

infinitely risk-averse agent: A_1 is bdd. from below by A_d.

$$A_0 = -\frac{1}{(1+r)\gamma}\ln\left(\mathbb{E}\left[e^{-\gamma(A_1-A_d)}\right]e^{-\gamma A_d}\right)$$

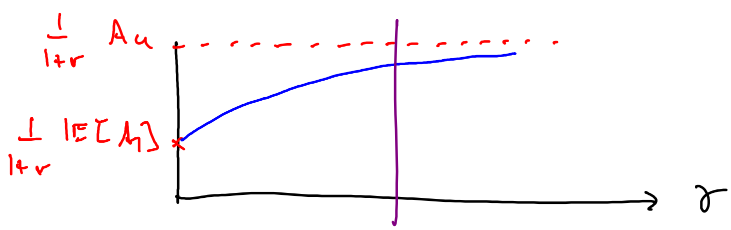
$$= \underline{\underline{\frac{1}{1+r}A_d}} - \underbrace{\frac{1}{(1+r)\gamma}\ln\mathbb{E}\left[e^{-\gamma(A_1-A_d)}\right]}_{\xrightarrow[\gamma\uparrow+\infty]{}0}$$



This was for the buyer ↑

For seller you can find

$$A_0 = +\frac{1}{(1+r)\gamma}\ln\mathbb{E}\left[e^{\gamma A_1}\right]$$



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Arbitrage : " riskless profit, in excess of the risk-free rate, through a self-financing strategy "

value of a self-financing strategy V_t

an arbitrage is a strategy s.t.

i) $V_0 = 0$ (costs nothing)

ii) $\exists$ a t s.t.

a. $\mathbb{P}(V_t \geq 0) = 1$ (never lose)

b. $\mathbb{P}(V_t > 0) > 0$ (sometimes win)

$$\begin{tikzcd}[row sep=small] & 10 \\ 0 \arrow[ur, dash] \arrow[dr, dash] & \\ & -1 \end{tikzcd}$$

is not an arb.

$$\begin{tikzcd}[row sep=small] & 2 \\ 1 \arrow[ur, dash] \arrow[dr, dash] & \\ & 1 \end{tikzcd}$$

is not an arb.

$$\begin{tikzcd}[row sep=small] & 1 \\ 0 \arrow[ur, dash] \arrow[dr, dash] & \\ & 0 \end{tikzcd}$$

is an arb.

$$\begin{tikzcd}[row sep=small] & 2 \\ 1 \arrow[ur, dash] \arrow[dr, dash] & \\ & 1 \end{tikzcd} \qquad \left. \begin{tikzcd}[row sep=small] & 1+r^{=0} \\ 1 \arrow[ur, dash] \arrow[dr, dash] & \\ & 1+r_{=0} \end{tikzcd} \right\} \qquad \begin{tikzcd}[row sep=small] & 1 \\ 0 \arrow[ur, dash] \arrow[dr, dash] & \\ & 0 \end{tikzcd}$$

short

is not an arb ... but economy does admit an arb.

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α-units $\qquad\qquad$ β-units

$$A_0 \left\langle \begin{array}{l} A_u \\ \\ A_d \end{array} \right. \qquad 1 \left\langle \begin{array}{l} 1+r \\ \\ 1+r \end{array} \right. \Bigg\} \qquad \alpha A_0 + \beta \left\langle \begin{array}{l} \alpha A_u + (1+r)\beta \\ \\ \alpha A_d + (1+r)\beta \end{array} \right.$$

$$(A_0, 1) \left\langle \begin{array}{l} (A_u, 1+r) \\ \\ (A_d, 1+r) \end{array} \right.$$

For an arb: $\qquad \alpha A_0 + \beta = V_0 = 0$

$$\Rightarrow \quad \beta = -\alpha A_0$$

to avoid arb:

$$0 \left\langle \begin{array}{lll} \alpha\left(A_u - (1+r)A_0\right) & > 0 & < 0 \\ & \quad\text{OR} & \\ \alpha\left(A_d - (1+r)A_0\right) & < 0 & > 0 \end{array} \right.$$

but $A_u > A_d$ so cannot happen!

$$A_u - (1+r)A_0 > 0 > A_d - (1+r)A_0$$

$\Rightarrow$ no arb.

$$\Rightarrow \quad \boxed{A_d < (1+r)A_0 < A_u}$$

$A_d \qquad (1+r)A_0 \qquad A_u$

recall from indifference pricing:

$$A_0 = -\frac{1}{(1+r)\gamma} \ln \mathbb{E}\left[e^{-\gamma A_1} \right]$$

For any finite r $\uparrow$ admits no arb!

Show that no arb $\Rightarrow$ $A_d < (1+r)A_0 < A_u$

There is no arb in economy iff

$$A_d < (1+r)\, A_0 < A_u$$

$$A_0 \begin{cases} A_u \\ A_d \end{cases} \qquad 1 \begin{cases} 1+r \\ 1+r \end{cases} \qquad B_0 \begin{cases} B_u \\ B_d \end{cases}$$

$$\left[(A_0, B_0, 1) \begin{cases} (A_u, B_u, 1+r) \\ (A_d, B_d, 1+r) \end{cases} \right]$$

$$B_0 \overset{?}{=} -\frac{1}{(1+r)\gamma} \ln \mathbb{E}\left[e^{-\gamma B_1} \right]$$

α - units of asset A
β - units of asset MM

$$\alpha A_0 + \beta \begin{cases} \underline{\alpha} A_u + (1+r) \underline{\beta} = B_u \\ \underline{\alpha} A_d + (1+r) \underline{\beta} = B_d \end{cases}$$

Since the portfolio replicates the value of asset B at $t=1$, the portfolio must be equal to asset B value at $t=0$. i.e.

$$B_0 = \alpha A_0 + \beta$$

otherwise there is an arbitrage!

$$10 \begin{cases} 20 \\ 5 \end{cases} \qquad 1 \begin{cases} 1 \\ 1 \end{cases} \qquad B_0 \begin{cases} 30 \\ 0 \end{cases}$$

α $\qquad$ β $\qquad$ $B_0 = ?$

i) find B_0 using replication

ii) suppose $B_0 = 25$ construct an arbitrage.

i)
$$20\alpha + \beta = 30$$
$$5\alpha + \beta = 0$$

$$15\alpha = 30 \Rightarrow \alpha = 2 \Rightarrow \beta = -10$$

$$B_0 = 10\alpha + \beta = 10$$

ii) sell B_1 buy 2 of A
sell 10 of MM

a)

$$25 - 2\times 10 + 10 = 15 \quad \left\langle \begin{matrix} 0 \\ 0 \end{matrix} \right.$$

buy 15 more of MM

$$0 \quad \left\langle \begin{matrix} 15 \\ 15 \end{matrix} \right.$$

$$B_0 = \frac{1}{1+r}\left(q\, B_u + (1-q)\, B_d\right) \stackrel{?}{=} \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[B_1]$$

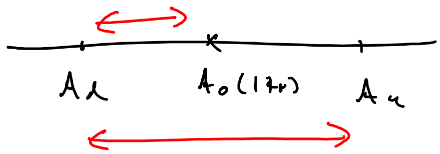
where
$$q = \frac{(1+r)A_0 - A_d}{A_u - A_d}$$

$$\mathbb{Q}(B_1 = B_u) = q \qquad \mathbb{P}(B_1 = B_u) = p$$

$$\mathbb{Q}(B_1 = B_d) = 1-q \qquad \mathbb{P}(B_1 = B_d) = 1-p$$

no arb $\iff A_d < (1+r) A_0 < A_u$

$\Rightarrow \quad 0 < q < 1 \qquad \Uparrow$



$$\boxed{q \in (0,1) \text{ iff there is no arb}}$$

$$\boxed{\begin{array}{l} \exists \text{ a } \mathbb{Q} \text{ s.t.} \\ B_0 = \frac{1}{1+r} \mathbb{E}^{\mathbb{Q}}[B_1] \\ \text{iff there is no arb.} \end{array}} \iff \mathbb{E}^{\mathbb{Q}}[B_1] = (1+r) B_0$$

$\mathbb{Q}$ is called the risk-neutral measure b/c <u>all</u> traded assets grow at risk-free under the risk-neutral measure!

$$A_0 \begin{cases} A_u & p_u \rightarrow q_u \\ A_m & p_m \rightarrow q_m \\ A_d & p_d \rightarrow q_d \end{cases}$$

s.t. $A_0(1+r) = q_u A_u + q_m A_m + q_d A_d$

$q_m, q_d, q_u > 0$

$q_m + q_d + q_u = 1$

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$$A \qquad\qquad MM$$

$$100 \left\langle \begin{array}{l} 110 \\ 50 \\ 0 \end{array} \right. \qquad 1 \left\langle \begin{array}{l} 1 \\ 1 \\ 1 \end{array} \right.$$

a) direct method: α - units of A
β - units of MM

$$V_0 = 0 \Rightarrow \beta = -100\,\alpha$$

$$0 \left\langle \begin{array}{l} 10\,\alpha \\ -50\,\alpha \\ -100\,\alpha \end{array} \right. \qquad \text{if } \mathbb{P}(V_1 \geq 0) = 1$$

$$\mathbb{P}(V_1 > 0) > 0 \quad ?$$

so if $\mathbb{P}(V_1 \geq 0) = 1 \Rightarrow \alpha = 0$

$$\Rightarrow \mathbb{P}(V_1 > 0) = 0$$

$\therefore$ this economy is arb. free!

b) Probabilistic method:

$$A_0 = \frac{1}{1+\underbrace{r}_{=0}} \mathbb{E}^{\mathbb{Q}}[A_1]$$

$$\overset{=1}{MM_0} = \frac{1}{1+\underbrace{r}_{=0}} \mathbb{E}^{\mathbb{Q}}[\overset{=1}{MM_1}]$$

$$100 = q_u\, 110 + q_m\, 50 + (1 - q_u - q_m)\, 0$$

$$= 110\, q_u + 50\, q_m$$

$$q_m = \frac{100 - 110 q_u}{50} = 2 - \frac{11}{5} q_u$$

$$0 < q_u < 1$$

$$0 < q_m < 1$$

$$\Rightarrow \quad 0 < 2 - \frac{11}{5} q_u < 1$$

$$\Rightarrow \quad 0 < 10 - 11 q_u < 5$$

$$\Rightarrow \quad q_u > \frac{5}{11}$$

$$q_u < \frac{10}{11}$$

$$\frac{5}{11} < q_u < \frac{10}{11}$$

$$0 < q_d < 1$$