Stochastic Methods for Actuarial Science - Problem Set #2
due Nov, 17 at 2pm.
Hand in only questions marked with **.
Grad students also hand in questions marked with ++

1. Suppose that W_t and Z_t are correlated Wiener processes with instantaneous correlation ρ. Find the mean and variance of each of the following random variables (T is fixed and positive):

   (a) $W_1 + W_2 + \ldots + W_N$ for $N \in \mathbb{N}$.

   (b) $\exp\{W_T\}$.

   (c) $a\,W_T + b\,Z_T$.

   (d) [**] $\exp\{a\,W_T + b\,Z_T\}$.

   (e) $W_T\,Z_T$.

   (f) [**] $\int_0^T W_s\,Z_s\,ds$.

2. Suppose that W_t and Z_t are standard correlated Wiener processes with instantaneous correlation ρ. Find the SDE which X_t solves:

   (a) $X_t = W_t^n$

   (b) $X_t = \cos(W_t)$

   (c) $X_t = W_t\,Z_t$

   (d) $X_t = \exp\{\alpha\,Z_t + \beta\,W_t\}$

3. In the following W_t and Z_t denote correlated Wiener processes with instantaneous correlation ρ. For each of the following (i) compute the mean and variance of Y_t and (ii) derive an integration by parts formulae:

   (a) $Y_t = \int_0^t W_s\,dW_s$.

   (b) $Y_t = \int_0^t (W_s)^2\,dW_s$.

   (c) $Y_t = \int_0^t s\,W_s\,dW_s$.

   (d) $Y_t = \int_0^t s\,W_s^2\,dW_s$.

   (e) $Y_t = \int_0^t s^2\,W_s^2\,dW_s$.

   (f) $Y_t = \int_0^t e^{-W_s}\,dW_s$.

   (g) [**] $Y_t = \int_0^t s\,e^{-W_s}\,dW_s$.

   (h) $Y_t = \int_0^t W_s\,dZ_s$.

   (i) [**] $Y_t = \int_0^t s\,W_s\,dZ_s$.

4. Prove the following from first principals, i.e. using the fundamental definition of the Ito integral (Note: W_t and Z_t denote correlated Wiener processes with instantaneous correlation ρ.):

   (a) $\int W_s^2 dW_s = \frac{1}{3}W_t^3 - \int_0^t W_s\,ds$

   (b) [++] $\int_0^t W_s\,dZ_s + \int_0^t Z_s\,dW_s = W_t\,Z_t - \rho\,t$

5. Assume that the prices of two stocks S_t and U_t satisfy the following coupled SDEs:

$$\frac{dU_t}{U_t} = a_t\,dt + b_t\,dW_t^U\ , \qquad \frac{dS_t}{S_t} = c_t\,dt + d_t\,dW_t^S \tag{1}$$

   where W_t^U and W_t^S denote two correlated Wiener processes with constant instantaneous correlation ρ and a_t, b_t, c_t, d_t are deterministic functions only of time t.

   (a) Compute each of the following:

      i. $d\ln(S_t)$

      ii. $d(S_t\,U_t)$

      iii. $d(S_t/U_t)$

      iv. [**] $d((S_t)^\alpha\,(U_t)^\beta)$ for $\alpha \neq 0; \beta \neq 0$.

   (b) Solve the system of SDEs (1).

   (c) What is the distribution of $Y = U_T S_T$ for a fixed $T > 0$?

6. Suppose X_t satisfies the SDE:

$$dX_t = \kappa(\theta - X_t)\,dt + \sigma\,dW_t\ , \qquad X_0 = x\ .$$

   show that the following is a solution

$$X_0 = \theta + (x - \theta)\,e^{-\kappa t} + \sigma \int_0^t e^{-\kappa(t-u)} dW_u$$