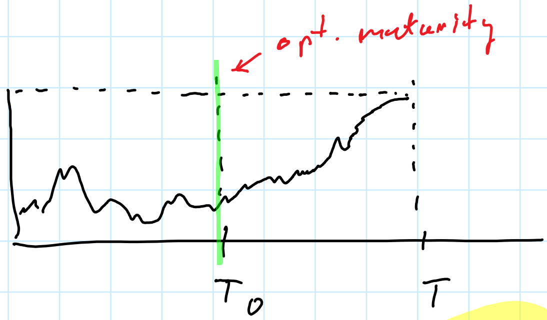
Bond Options: call option $(P_{T_0}(T) - K)_+$



$P_t(T) = e^{A(t) - C(t) r_t}$

$e^{A - C r_{T_0}}$

$$V_t = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_0} r_u du} \left(P_{T_0}(T) - K \right)_+ \right]$$

$$\begin{pmatrix} \int_t^{T_0} r_u \\ r_{T_0} \end{pmatrix} \underset{\mathbb{Q}}{\sim} \mathcal{N}\left(\begin{pmatrix} \; \\ \; \end{pmatrix} ; \begin{pmatrix} \; & \; \\ \; & \; \end{pmatrix} \right)$$

$\mathbb{E}\left[\int h_s dW_s^{\mathbb{Q}} \int g_s dW_s^{\mathbb{Q}} \right]$

$= \mathbb{E}\left[\int h_s g_s ds \right]$

Can change measure and use the bond as a numeraire asset:

$$\frac{V_t}{P_t(T_0)} = \mathbb{E}_t^{\mathbb{Q}^{T_0}}\left[\frac{(P_{T_0}(T) - K)_+}{P_{T_0}(T_0)} \right]$$

$P_{T_0}(T_0) \to 1$

numeraire induced by T_0-bond

(forward-neutral measure)

$$V_t = P_t(T_0)\, \mathbb{E}_t^{\mathbb{Q}^{T_0}}\left[\left(P_{T_0}(T) - K \right)_+ \right]$$

$\hookrightarrow e^{A - C r_{T_0}}$

$\hookrightarrow \underset{\mathbb{Q}}{\sim} \mathcal{N}(\diamond ; \cdot)$

$\Rightarrow \underset{\mathbb{Q}_{T_0}}{\sim} \mathcal{N}(\square ; \cdot)$

introduce

$$X_t := \frac{P_t(T)}{P_t(T_0)} \xrightarrow[t \uparrow T_0]{} \frac{P_{T_0}(T)}{P_{T_0}(T_0)} = P_{T_0}(T)$$

$$\Rightarrow \quad V_t = P_t(T_0)\, \mathbb{E}_t^{\mathbb{Q}^{T_0}}\left[\left(X_{T_0} - K \right)_+ \right]$$

Note: since $X = \dfrac{T\text{-bond price}}{T_0\text{-bond price}}$ it is

a $\mathbb{Q}_{T_0}$-martingale.

$$dX_t = (\quad)\,dt + (\quad)\,dW_t^{\mathbb{Q}}$$

$$= (0)\,dt + (\quad)\,dW_t^{\mathbb{Q}^{T_0}}$$

$\hookrightarrow$ b/c of the $\mathbb{Q}_{T_0}$-mtg property of X

$$P_t(T) = e^{A_t(T) - B_t(T)\, r_t}$$

$$P_t(T_0) = e^{A_t(T_0) - B_t(T_0)\, r_t}$$

$$X_t = e^{\Delta A_t - \Delta B_t\, r_t} = f(t, r_t)$$

where $f(t,r) = e^{\Delta A_t - \Delta B_t\, r}$

from Ito's Lemma

$$d X_t = (\quad) dt + \underbrace{\partial_r f(t, r_t)}_{-\Delta B_t f(t, r_t) \;\hookrightarrow\; X_t} \cdot \sigma \, dW_t^{\mathbb{Q}}$$

$$\Rightarrow \quad d X_t = - X_t \, \Delta B_t \, \sigma \, dW_t^{\mathbb{Q}_{T_0}}$$

NB: X is a GBM with zero drift but deterministic volatility.

$$\Rightarrow \quad X_{T_0} = X_t \exp\left\{ - \frac{1}{2} \int_t^{T_0} \Delta B_u^2 \, \sigma^2 \, du - \int_t^{T_0} \Delta B_u \, \sigma \, dW_u^{\mathbb{Q}_{T_0}} \right\}$$

recall that

$$V_t = P_t(T_0) \, \mathbb{E}^{\mathbb{Q}_{T_0}} \left[(X_{T_0} - K)_+ \right]$$

recall in B-S (when $r = 0$) we have

$$\mathbb{E}^{\mathbb{Q}} \left[(S_{T_0} - K)_+ \right] = \ldots$$

$$S_{T_0} = S_t \, e^{-\frac{1}{2}\sigma^2 (T_0 - t) + \sigma (W_{T_0} - W_t)}$$

$$\stackrel{d}{=} S_t \, e^{-\frac{1}{2} \sigma^2 (T_0 - t) + \sigma \sqrt{T_0 - t} \, Z}, \quad Z \underset{\mathbb{Q}}{\sim} N(0,1)$$

$$X_{T_0} \stackrel{d}{=} X_t \, e^{-\frac{1}{2}\Sigma^2 + \Sigma Y}, \quad Y \underset{\mathbb{Q}_{T_0}}{\sim} N(0,1)$$

$$\Sigma^2 = \sigma^2 \int_t^{T_0} \Delta B^2$$

$\mathbb{Q}^{T_0}$

where $\Sigma^2 = \sigma^2 \int_t^{T_0} \Delta B_u^2 \, du$

↳ total variance of $\int_t^{T_0} \sigma \, \Delta B_u \, dW_u^{\mathbb{Q}^{T_0}}$

Now apply B-S with $r = 0$ and total variance Σ^2

$$\Rightarrow \quad V_t = P_t(T_0) \, \mathbb{E}^{\mathbb{Q}^{T_0}}\left[(X_{T_0} - K)_+ \right]$$

$$= P_t(T_0) \left\{ X_t \, \Phi(d_+) - K \, \Phi(d_-) \right\}$$

$$d_\pm = \frac{\log(X_t / K) \pm \frac{1}{2}\Sigma^2}{\Sigma}$$

$$\boxed{ V_t = P_t(T) \, \Phi(d_+) - K P_t(T_0) \, \Phi(d_-) \qquad d_\pm = \frac{\log\left(\frac{P_t(T)}{K P_t(T_0)} \right) \pm \frac{1}{2}\Sigma^2}{\Sigma} }$$

¿ How about equity options when IR are stochastic?

$$V_t = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_u\,du} (S_T - K)_+ \right]$$

$$dr_t = \kappa(\theta - r_t)\,dt + \sigma\, dW_t^r$$

ρ

$$dS_t = S_t(r_t\,dt + \eta\, dW_t^S)$$

$$\frac{V_t}{P_t(T)} = \mathbb{E}_t^{\mathbb{Q}_T}\left[\frac{(S_T - K)_+}{P_T(T)} \right]$$

$\hookrightarrow 1$

$$X_t = \frac{S_t}{P_t(T)} \text{ is a } \mathbb{Q}_T\text{-mtg.}$$

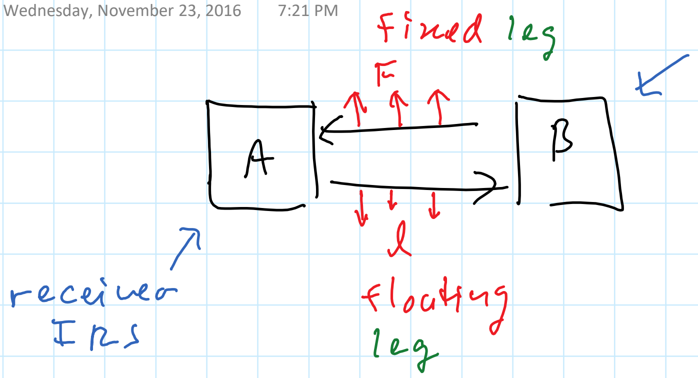
$$\xrightarrow[t \uparrow T]{} S_T$$

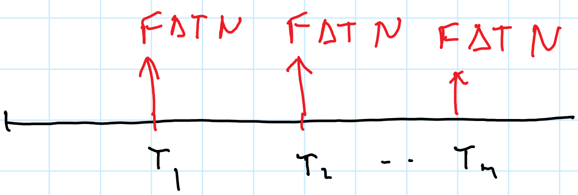
$$\Rightarrow V_t = P_t(T)\, \mathbb{E}^{\mathbb{Q}_T}\left[(X_T - K)_+ \right]$$

$$dX_t = \ldots$$

Interest Rate Swap (IRS)

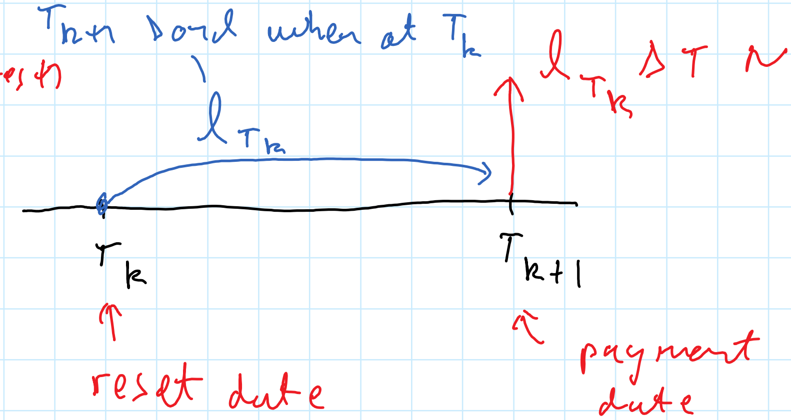
Wednesday, November 23, 2016 7:21 PM

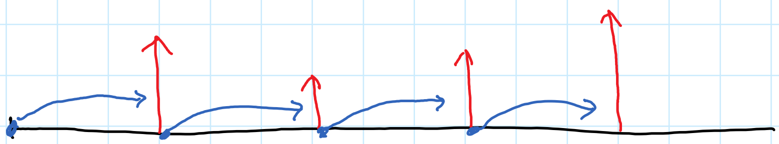




applied to a notional N
reported as annualized
simple interest rate

"yield" of T_{k+1} bond when at T_k
(simple interest)

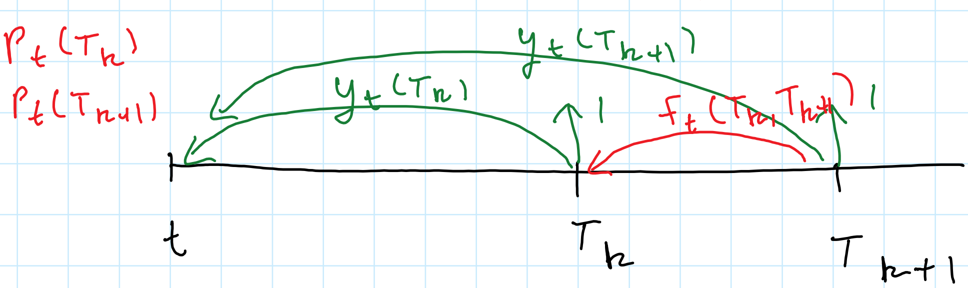




LIBOR

$$P_{T_k}(T_{k+1}) := (1 + \Delta T\, l_{T_k})^{-1}$$

$$= e^{-\Delta T\, \underline{y_{T_k}(T_{k+1})}}$$



$$P_t(T_{k+1}) = \left[1 + \Delta T \, f_t(T_k, T_{k+1})\right]^{-1} P_t(T_k)$$

forward rate of interest $[T_k, T_{k+1}]$

$$\boxed{\begin{array}{l} l_{T_k} \text{ is } \mathcal{F}_{T_k}\text{-measurable} \\ f_t(T_k, T_{k+1}) \text{ is } \mathcal{F}_t\text{- adapted} \end{array}}$$

$$\boxed{V_t^{fix} = F \, \Delta T \, N \sum_{k=1}^{n} P_t(T_k)}$$

$$V_t^{Fl,k} = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_{k+1}} r_u \, du} \cdot l_{T_k} \Delta T \, N \right]$$

NB: $l_{T_k} = \frac{1}{\Delta T}\left[\frac{1}{P_{T_k}(T_{k+1})} - 1\right]$

$$\frac{V_t^{Fl,k}}{P_t(T_{k+1})} = \mathbb{E}_t^{\mathbb{Q}^{T_{k+1}}}\left[\frac{\left(\frac{1}{P_{T_k}(T_{k+1})} - 1\right) N}{P_{T_{k+1}}(T_{k+1})}\right]$$

$\hookrightarrow 1$

$$\Rightarrow \quad V_t^{fl,k} = P_t(T_{k+1}) \, \mathbb{E}_t^{\mathbb{Q}^{T_{k+1}}}\left[\left(\frac{1}{P_{T_k}(T_{k+1})} - 1\right)\right] N$$

$$X_t \triangleq \frac{P_t(T_k)}{P_t(T_{k+1})} \quad \text{is a } \mathbb{Q}^{T_{n+1}}\text{-mtg}$$

$$\xrightarrow[t \uparrow T_k]{} \quad \frac{1}{P_{T_k}(T_{k+1})}$$

NB: $$\mathbb{E}_t^{\mathbb{Q}^{T_{k+1}}}\left[X_{T_k} \right] = X_t$$

Hence $$V_t^{fl,k} = P_t(T_{k+1}) \, (X_t - 1) \, N$$

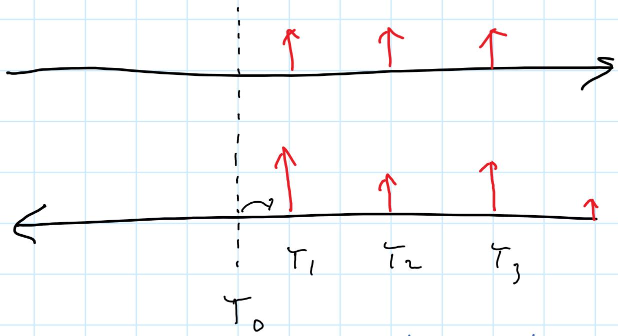
$$\boxed{V_t^{fl,k} = \left(P_t(T_k) - P_t(T_{k+1})\right) N}$$

$$V_t^{fl} = \sum_{k=0}^{n-1} V_t^{fl,k}$$

$$\boxed{V_t^{fl} = \left(P_t(T_0) - P_t(T_n)\right) N}$$

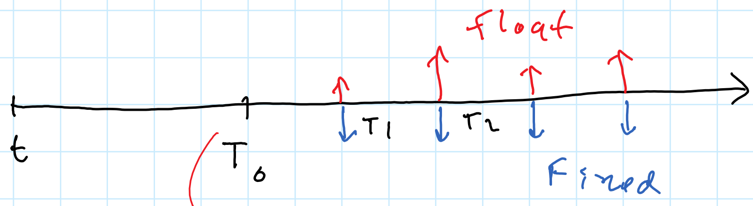
Swap-rate S_t' is the fixed rate F which makes $V_t^{fl} = V_t^{fix}$

$$\boxed{S_t' = \frac{P_t(T_0) - P_t(T_n)}{\sum_{k=1}^{n} P_t(T_k)\,\Delta T}}$$



↳ option to enter payer leg at F

swap - option , swaption



option to
enter the payer IRS at rate F

if $S_{T_0} > F$, then enter

$S_{T_0} \le F$, otherwise don't

fixed-leg with rate "1"

$$V_{T_0} = \left(V^{fl}_{T_0} - F \, \overline{V}^{fix}_{T_0} \right)_+$$

$$= \left(\frac{V^{fl}_{T_0}}{V^{fix}_{T_0}} - F \right)_+ \overline{V}^{fix}_{T_0}$$

$$= \left(S_{T_0} - F \right)_+ \overline{V}^{fix}_{T_0}$$

$$V_t = \mathbb{E}^{\mathbb{Q}}_t \left[e^{-\int_t^T r_u \, du} \left(S_{T_0} - F \right)_+ \overline{V}^{fix}_{T_0} \right]$$

$$\overline{V}^{fix}_t = \sum_{k=1}^{n} P_t(T_k) \, \Delta T \quad \text{(annuity)}$$

$$\frac{V_t}{\overline{V}^{fix}_t} = \mathbb{E}^{\mathbb{Q}^A}_t \left[\frac{\left(S_{T_0} - F \right)_+ \overline{V}^{fix}_{T_0}}{\overline{V}^{fix}_{T_0}} \right]$$

$$\Rightarrow \quad V_t = \overline{V}^{fix}_t \, \mathbb{E}^{\mathbb{Q}^A}_t \left[\left(S_{T_0} - F \right)_+ \right]$$

recall: $$S_t = \frac{P_t(T_0) - P_t(T_n)}{\sum_{k=1}^{n} P_t(T_k)\,\Delta T}$$

is a $\mathbb{Q}^A$ - mtg !

Hence $\exists\, \sigma_t^{S}$ s.t. $\frac{dS_t}{S_t} = \sigma_t^{S}\, dW_t^{\mathbb{Q}_A}$

↳ if this is deterministic the model is called lognormal - swap - rate model (LSM)

under LSM ⟶ apply B-S like formula to value swaption.

recall

$$V_{T_0} = \left(V_{T_0}^{fl} - V_{T_0}^{fix}\right)_+$$

$$= \left(\left(P_{T_0}(T_0) - P_{T_0}(T_n)\right) - \sum_{k=1}^{n} P_{T_0}(T_k)\, F\,\Delta T\right)_+$$

$$= \left(1 - \sum_{k=1}^{n} c_k\, P_{T_0}(T_k)\right)_+$$

$$c_k = \begin{cases} F\,\Delta T, & k \neq n \\ 1 + F\,\Delta T, & k = n \end{cases}$$

NB: if $P_{T_0}(T_k) = e^{A_k - B_k r_{T_0}} = f_k(r_{T_0})$

as $r_{T_0} \nearrow \quad P_{T_0}(T_k) \searrow \quad \forall k$

$\Rightarrow \exists !\ r^* \quad s.t.$

$$\sum_{k=1}^{n} c_k F_k(r^*) = 1$$

$\Rightarrow$

$$V_{T_0} = \left(\sum_{k=1}^{n} c_k \left(f_k(r^*) - P_{T_0}(T_k) \right) \right) \mathbb{1}_{r_{T_0} > r^*}$$

$$= \sum_{k=1}^{n} c_k \left(f_k(r^*) - P_{T_0}(T_k) \right) \mathbb{1}_{r_{T_0} > r^*}$$

$$\mathbb{1}_{r_{T_0} > r^*} = \mathbb{1}_{-r_{T_0} < -r^*}$$

$$= \mathbb{1}_{-B_k r_{T_0} < -B_k r^*}$$

$$= \mathbb{1}_{A_k - B_k r_{T_0} < A_k - B_k r^*}$$

$$= \mathbb{1}_{e^{A_k - B_k r_{T_0}} < e^{A_k - B_k r^*}}$$

$$= \mathbb{1}_{P_{T_0}(T_k) < f_k(r^*)}$$

$$\Rightarrow V_{T_0} = \sum_{k=1}^{M} c_k \left(f_k(r^*) - P_{T_0}(T_k) \right) \mathbb{1}_{P_{T_0}(T_k) < f_k(r^*)}$$

$$= \sum_{k=1}^{n} c_k \left(f_k(r^*) - P_{T_0}(T_k) \right)_+$$

This is a strip of put options on T_k bonds with strikes $f_k(r^*)$

$$\Rightarrow \quad V_t = \sum_{k=1}^{M} c_k \, V_t^{put}\big(\text{strike} = f_k(r^*), \text{ bond underlying mat } T_k\big)$$

option mat T_0