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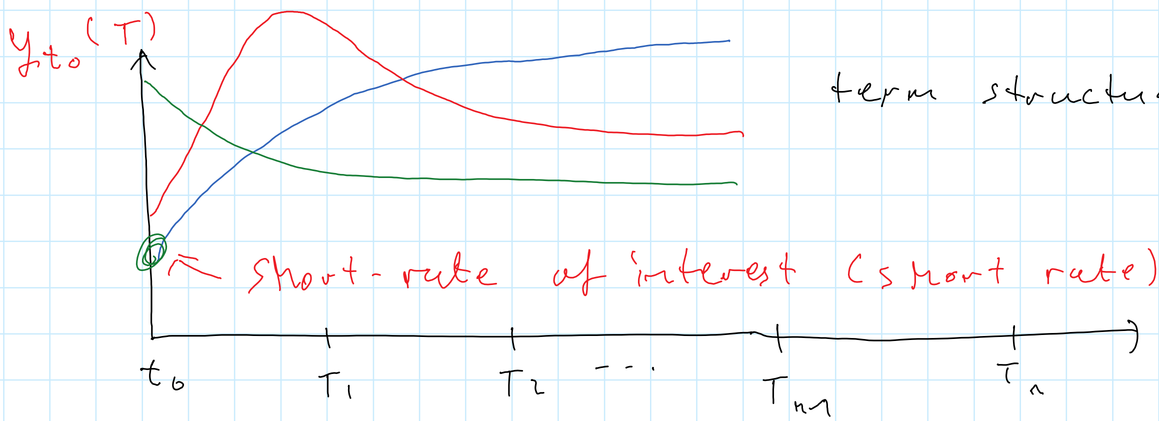
FTAP: no arb $\iff$ $\exists\ \mathbb{Q}^c \sim \mathbb{P}$, s.t. relative prices of traded assets are martingales.
(relative to some numeraire c)

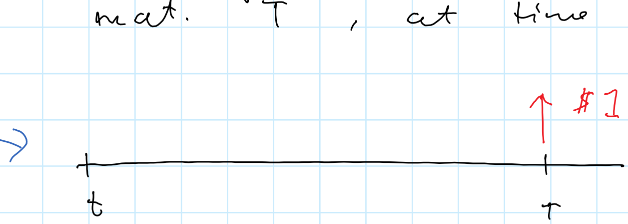
Interest Rates (IR)



term structure of IR

$y_t(T)$ - yield of a bond of maturity T viewed at t.

$e^{-y_t(T)(T-t)} = P_t(T)$ - price of zero-coupon bond mat. T, at time t.



$$B_{t_k} = B_{t_{k-1}}\left(1 + r_{t_{k-1}} \Delta t_k\right)$$

$$\rightarrow B_{t_n} = B_0 \prod_{k=1}^{n} \left(1 + r_{t_{k-1}} \Delta t_k\right)$$

$\log(1+x) \sim x$

$$= B_0 \exp\left\{ \sum_{k=1}^{n} \log\left(1 + r_{t_{k-1}} \Delta t_k\right) \right\}$$

$$\longrightarrow B_t = \exp\left\{ \int^t r \ldots \right\}$$

$k=1$

$$\xrightarrow{\|\Pi\|\downarrow 0} \quad B_0 \exp\left\{ \int_0^t r_u \, du \right\}$$

Some models of IR:

$$dr_t = \sigma(t)\, dW_t^{\mathbb{P}} \qquad \text{Ho - Lee}$$

OU : $dr_t = \kappa(\theta - r_t)\, dt + \sigma\, dW_t^{\mathbb{P}}$ Vasicek

Feller: $dr_t = \kappa(\theta - r_t)\, dt + \sigma \sqrt{r_t}\, dW_t^{\mathbb{P}}$ Cox Ingersoll Ross (CIR)

relative bond price: $\dfrac{P_t(T)}{B_t}$ is a $\mathbb{Q}$-mtg

$$\Rightarrow \quad \frac{P_t(T)}{\underbrace{B_t}_{e^{\int_0^t r_u du}}} = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{\overbrace{P_T(T)}^{1}}{\underbrace{B_T}_{e^{\int_0^T r_u du}}} \right], \qquad t \le T$$

$$P_t(T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_u du} \right]$$

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}\left(-\int_0^T \lambda_s \, dW_s^{\mathbb{P}} \right)$$

$W_t^{\mathbb{Q}} = \int_0^t \lambda_s \, ds + W_t^{\mathbb{P}}$ is a $\mathbb{Q}$ - B.mtn

$$W_t^{\mathbb{Q}} = \int_0^t \lambda_s \, ds + W_t^{\mathbb{P}} \quad \text{is a } \mathbb{Q}\text{-B.mtn}$$

$$\Rightarrow \quad dr_t = \mu^r(t, r_t)\, dt + \sigma^r(t, r_t)\, dW_t^{\mathbb{P}}$$

$$= \underbrace{\left(\mu^r(t, r_t) - \lambda(t, r_t)\, \sigma^r(t, r_t)\right)}_{\mu^{r\,\mathbb{Q}}} dt$$

$$+ \sigma^r(t, r_t)\, dW_t^{\mathbb{Q}}$$

Vasicek: $\mu^r(t, r) = \kappa(\theta - r)$, $\quad \sigma^r(t, r) = \sigma$

choose $\lambda(t, r) = \lambda_0 + \lambda_1 r$

$$\mu^{r,\mathbb{Q}} = \kappa\theta - \kappa r - \lambda_0 \sigma - \lambda_1 \sigma r$$

$$= (\kappa\theta - \lambda_0\sigma) - (\kappa + \lambda_1\sigma)\, r$$

$$= \hat{\kappa}(\hat{\theta} - r)$$

$$\hat{\kappa} = \kappa + \lambda_1 \sigma, \quad \hat{\theta} = \frac{\kappa\theta - \lambda_0\sigma}{\kappa + \lambda_1\sigma}$$

$$P_t(T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_u \, du} \right] = f\left(t, r_t, \int_0^t r_u \, du\right)$$

$$dr_t = \kappa(\theta - r_t)\, dt + \sigma\, dW_t^{\mathbb{Q}}$$

1. compute distribution of $\int_t^T r_u \, du \,\Big|_{\mathcal{F}_t}$
2. Feynman-Kac + solve PDE.

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$$dr_t = \kappa(\theta - r_t)\,dt + \sigma\,dW_t^{\mathbb{Q}}$$

$$d(e^{+\kappa t} r_t) = +\kappa e^{+\kappa t} r_t\,dt + e^{+\kappa t}\,dr_t + \underbrace{d[e^{+\kappa\cdot}, r]_t}_{\to 0}$$

$$= +\kappa e^{+\kappa t} r_t\,dt + e^{+\kappa t}\left[\kappa(\theta - r_t)\,dt + \sigma\,dW_t^{\mathbb{Q}}\right]$$

$$= \kappa\theta e^{+\kappa t}\,dt + e^{+\kappa t}\sigma\,dW_t^{\mathbb{Q}}$$

$u > t$, ... $\int_t^u (\quad) \Rightarrow$

$$e^{\kappa u} r_u - e^{\kappa t} r_t = \theta(e^{\kappa u} - e^{\kappa t}) + \sigma\int_t^u e^{\kappa s}\,dW_s^{\mathbb{Q}}$$

$$\boxed{r_u = r_t e^{-\kappa(u-t)} + \theta(1 - e^{-\kappa(u-t)}) + \sigma\int_t^u e^{-\kappa(u-s)}\,dW_s^{\mathbb{Q}}}$$

[$r_t = e^{\pm\kappa t} g_t \longrightarrow$ find SDE for g_t

$r_t = h_t + g_t \longrightarrow$ find both h_t & g_t s.t. can integrate]

$$r_u\big|_{\mathcal{F}_t} \underset{\mathbb{Q}}{\sim} \mathcal{N}\Big(\underbrace{(r_t - \theta)e^{-\kappa(u-t)} + \theta}_{\xrightarrow{u\to\infty}\ \theta}\ ;\ \Sigma_r^2(t,u)\Big)$$

$$\Sigma_r^2(t,u) = \sigma^2\,\mathbb{E}\left[\left(\int_t^u e^{-\kappa(u-s)}\,dW_s^{\mathbb{Q}}\right)^2\right]$$

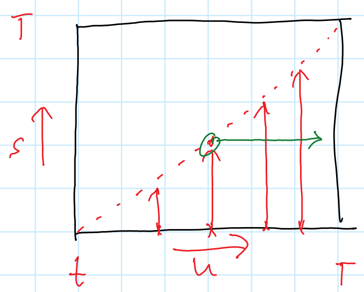
$$= \sigma^2\,\mathbb{E}\left[\int_t^u e^{-2\kappa(u-s)}\,ds\right]$$

$$= \sigma^2 \, \frac{1 - e^{-2\kappa(u-t)}}{2\kappa} \xrightarrow[u \uparrow \infty]{} \frac{\sigma^2}{2\kappa}$$

invariant distribution of r is $\mathcal{N}\left(\theta \,;\, \frac{\sigma^2}{2\kappa}\right)$

$$\int_t^T r_u \, du = (r_t - \theta) \overbrace{\int_t^T e^{-\kappa(u-t)} \, du}^{(1-e^{-\kappa(T-t)})/\kappa} + \theta (T-t)$$

$$+ \sigma \int_t^T \left(\int_t^u e^{-\kappa(u-s)} \, dW_s^{\mathbb{Q}} \right) du$$



$$\int_t^T \int_t^u e^{-\kappa(u-s)} dW_s^{\mathbb{Q}} \, du$$

$$= \int_t^T \underbrace{\int_s^T e^{-\kappa(u-s)} du}_{\frac{1 - e^{-\kappa(T-s)}}{\kappa}} \, dW_s^{\mathbb{Q}}$$

$$\left. \int_t^T r_u \, du \right|_{\mathcal{F}_t} \underset{\mathbb{Q}}{\sim} \mathcal{N}\left((r_t - \theta) \frac{(1 - e^{-\kappa(T-t)})}{\kappa} + \theta (T-t) \,; \right.$$

$$\left. \Omega^r(t,T) \right)$$

$$\Omega^r(t,T) = \sigma^2 \mathbb{E}_t^{\mathbb{Q}} \left[\left(\int_t^T \frac{1 - e^{-\kappa(T-s)}}{\kappa} dW_s^{\mathbb{Q}} \right)^2 \right]$$

$$= \frac{\sigma^2}{\kappa^2} \int_t^T \left(1 - e^{-\kappa(T-s)}\right)^2 ds$$

$$dr_t = \kappa(\theta - r_t) \, dt + \sigma \, dW_t^{\mathbb{Q}}$$

$$r_T - r_t = \kappa \theta (T-t) - \kappa \underbrace{\int_t^T r_u \, du} + \sigma \int_t^T dW_u^{\mathbb{Q}}$$

?

$$\Rightarrow \int_t^T r_u\,du = \theta(T-t) - (r_T - r_t)\frac{1}{\kappa} + \frac{\sigma}{\kappa}\int_t^T dW_u^{\mathbb{Q}}$$

$$= \theta(T-t) - \left\{ (r_t-\theta)\, e^{-\kappa(T-t)} + \theta + \sigma\int_t^T e^{-\kappa(T-u)}\,dW_u^{\mathbb{Q}} - r_t \right\}\frac{1}{\kappa}$$

$$+ \frac{\sigma}{\kappa}\int_t^T dW_u^{\mathbb{Q}} -$$

$$\boxed{\begin{aligned}\int_t^T r_u\,du = \theta(T-t) - \frac{(r_t-\theta)\, e^{-\kappa(T-t)} + (\theta - r_t)}{\kappa} \\ + \frac{\sigma}{\kappa}\int_t^T \left(1 - e^{-\kappa(T-u)}\right) dW_u^{\mathbb{Q}}\end{aligned}}$$

$$\left.\int_t^T r_u\,du\right|_{\mathcal{F}_t} \underset{\mathbb{Q}}{\sim} \mathcal{N}\left(a_t + b_t r_t \,;\, \Omega^r_t \right)$$

$$P_t(T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_u\,du} \right]$$

$$= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\left(a_t + b_t r_t + \sqrt{\Omega^r_t}\, Z\right)} \right]$$

$$Z \underset{\mathbb{Q}}{\sim} \mathcal{N}(0,1)$$

$$= e^{-(a_t + b_t r_t) + \frac{1}{2}\Omega^r_t}$$

$$= e^{A_t - C_t r_t}$$

affine model of interest rates

$$\boxed{P_t(T) = e^{A_t - C_t r_t}}$$

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2) Feynman-Kac

$$P_t(T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s\, ds} \right] = f(t, r_t)$$

$$dr_t = \kappa(\theta - r_t)\,dt + \sigma\, dW_t^{\mathbb{Q}}$$

for CIR $\sqrt{r_t}$

$$f : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}$$
$$t \quad r \quad p$$

$$\begin{cases} \partial_t f + \mathcal{L}^r f = r f \\ f(T, r) = 1 \end{cases}$$

$$\mathcal{L}^r = \kappa(\theta - r)\,\partial_r + \tfrac{1}{2}\sigma^2 \, r \, \partial_{rr}$$

ansatz $\quad f(t, r) = e^{A_t - C_t r}$

unknown deterministic fn of t

and terminal condition

ie $C_T = A_T = 0$

s/c $f(T, r) = 1 \;\; \forall r.$

$$\partial_t f = (\dot{A} - \dot{C}\, r)\, f$$

$\dot{A} = \frac{d}{dt}A \qquad \dot{C} = \frac{d}{dt}C$

$$\partial_r f = -C f \quad , \quad \partial_{rr} f = C^2 f$$

$$\Rightarrow \quad (\dot{A} - \dot{C} r) f + \kappa(\theta - r)(-C f) + \tfrac{1}{2}\sigma^2 C^2 f = r f$$

$$\underbrace{\left[\dot{A} + \kappa\theta + \tfrac{1}{2}\sigma^2 C^2\right]}_{0 \;\forall t} - \underbrace{(\dot{C} - \kappa C + 1)}_{0 \;\forall t}\, r = 0 \quad (*)$$

since (*) must hold $\forall\, t, r$

$$\dot{A} + \kappa\theta + \tfrac{1}{2}\sigma^2 C^2 = 0$$

$$A(T) - A(t) + \kappa\,\theta\,(T-t) + \tfrac{1}{2}\sigma^2 \int_t^T C^2(u)\,du = 0$$

0

$$\dot{C} - \kappa C + 1 = 0, \quad C(T) = 0$$

$$C(t) = \frac{1 - e^{-\kappa(T-t)}}{\kappa}$$

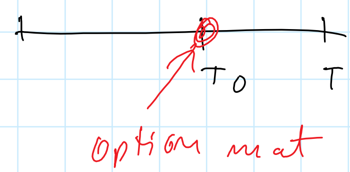
even in extended Vasicek / CIR models,

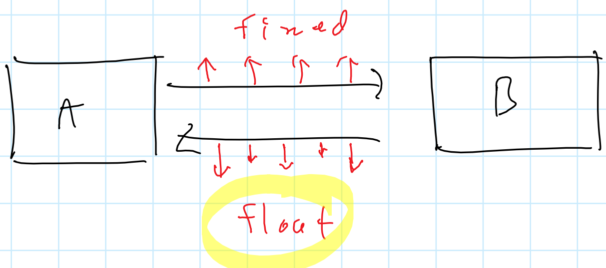
$$dr_t = \kappa(\theta_t - r_t)\,dt + \sigma\sqrt{r_t}\,dW_t^{\mathbb{Q}}$$

bond prices are affine: and depend on the function θ

$$\exists\ \theta \quad \text{s.t.} \quad P_t^{mdl}(T) = P_t^{data}(T)$$

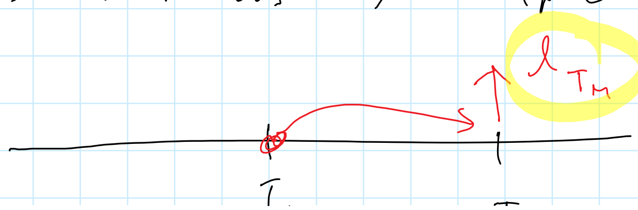
$$\left(P_{T_0}(T) - K\right)_+ \qquad \text{call on a bond}$$





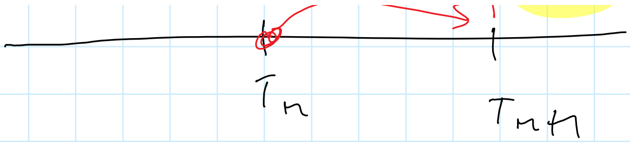
Interest Rate Swap (IRS)

Caps / Floors, Caplets / floorlets



$$\longrightarrow \quad (\ell_{T_m} - K)_+$$

$$(K - \ell_{T_m})_+$$



$(K - L_{T_n})_+$

Swaptions are an option to enter into a swap at a later date.