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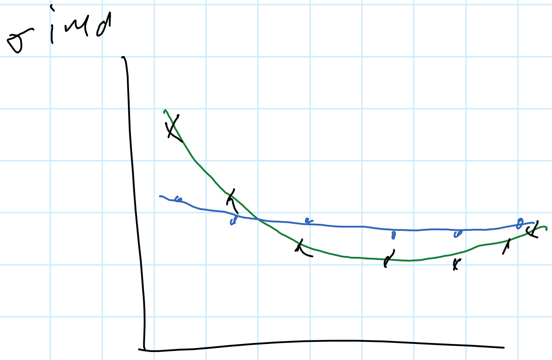
$$dS_t = S_t(\mu\, dt + \sigma(t, S_t)\, dW_t)$$

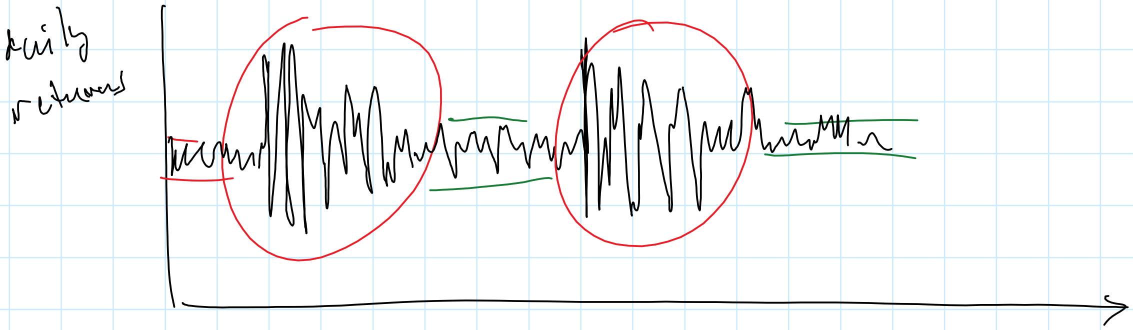


$S \downarrow \quad \sigma \uparrow$ leverage effect

$\sigma(t, S) = S^\gamma$ constant elasticity of variance CEV

$\sigma(t, S)$ local volatility model

stochastic volatility has a second source of uncertainty



e.g. Heston Model

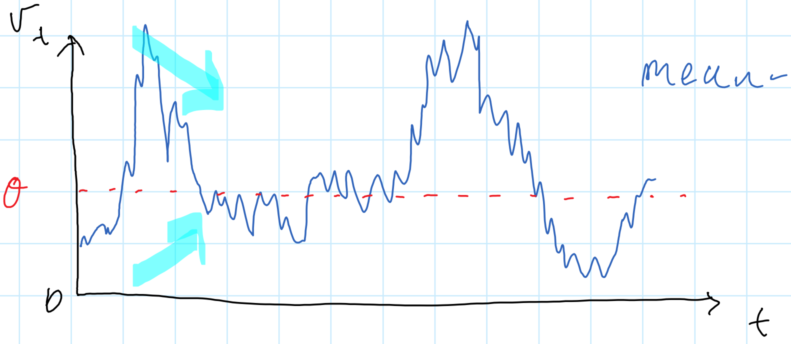
$$dS_t = S_t(\mu\, dt + \sqrt{v_t}\, dW_t^S)$$

$$dv_t = \kappa(\theta - v_t)\, dt + \eta\sqrt{v_t}\, dW_t^v$$ (Feller process)

$(\theta > \eta^2/2\kappa \;,\; \kappa, \eta, \theta > 0)$

CIR process
Cox, Ingersoll, Ross

$$d[W^S, W^v]_t = \rho \, dt$$



mean-reverting

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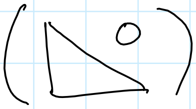
i) $X = ((X_t^{(1)}, X_t^{(2)}, \dots X_t^{(n)})')_{t\geq 0}$

$\mu^X(t,x)$, $\mu^X : \mathbb{R}_+ \times \mathbb{R}^n \longmapsto \mathbb{R}^n$

$\sigma^X(t,x)$, $\sigma^X : \mathbb{R}_+ \times \mathbb{R}^n \longmapsto \mathbb{R}^n \times \mathbb{R}^n$



$W = ((W_t^{(1)}, W_t^{(2)}, \dots, W_t^{(n)}))_{t\geq 0}$ are independent $\mathbb{P}$-B. mtn

$$\underset{(n\times 1)}{dX_t} = \underset{(n\times 1)}{\mu^X(t,X_t)}\,dt + \underset{(n\times n)}{\sigma^X(t,X_t)}\,\underset{(n\times 1)}{d\overset{W,\mathbb{P}}{W}_t}$$

$$\begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} ; \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right)$$

$$Y^1 = Z^1$$

$$Y^2 = \rho Z^1 + \sqrt{1-\rho^2}\, Z^2$$

Z^1, Z^2 are iid $\mathcal{N}(0,1)$

$$\begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} \begin{pmatrix} Z^1 \\ Z^2 \end{pmatrix}$$

- $B = (B_t)_{t\geq 0}$

$$dB_t = r(t, X_t)\, B_t\, dt$$

$$r : \mathbb{R}_+ \times \mathbb{R}^n \longmapsto \mathbb{R}$$

- $f = \left((f_t^{(1)}, \dots, f_t^{(n)}) \right)_{t \geq 0}$

$W^{\mathbb{P}}$

$$df_t^{(i)} = f_t^{(i)} \Big(\underbrace{\mu_i^f (t, X_t)}_{(1 \times 1)}\, dt + \underbrace{\sigma_i^f (t, X_t)}_{(1 \times n)}\, d\boxed{W}_t \Big)$$

ith row

$$\mu^f : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}^n$$

ith row

$$\sigma^f : \mathbb{R}_+ \times \mathbb{R} \longmapsto \mathbb{R}^n \times \mathbb{R}^n$$

value a claim with price process

$g_i = ((g_{i,t})_{t \geq 0})$ $\qquad$ $g_T = G_i(X_T)$

$$d g_t = g_t \Big(\underbrace{\mu_i^g (t, X_t)}_{(1 \times 1)}\, dt + \underbrace{\sigma_i^g (t, X_t)}_{(1 \times n)}\, dW_t \Big)$$

$$\frac{\mu_t^f - r_t}{\sigma_t^f} = \frac{\mu_t^g - r_t}{\sigma_t^g}$$

$$(\sigma_t^f)^{-1} (\mu_t^f - r_t) = (\sigma_t^g)^{-1} (\mu_t^g - r_t) = \lambda_t$$

property of the market.

$$\begin{cases} (\partial_t + \mathcal{L}_t^X)\, g(t, x) = r(t, x)\, g(t, x) \\ g(T, x) = G(x) \end{cases}$$

$\mathcal{L}^X_t$ infinitesimal generator of X under the $\mathbb{Q}$-measure.

$$\mathcal{L}^X_t\, g(t,x) = \lim_{s \downarrow t} \frac{\mathbb{E}[\, g(t, X_s) - g(t,x) \mid X_t = x \,]}{s-t}$$

$$\begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \vdots \end{pmatrix}$$

$$= \left(\mu^X(t,x) - \sigma^X(t,x)\, \lambda(t,x) \right)' \cdot D_x\, g(t,x)$$

$$+ \frac{1}{2}\, \mathrm{Tr}\left(\sigma^X(t,x)\, \sigma^X(t,x)'\, D_{xx}\, g(t,x) \right)$$

$$\begin{pmatrix} \partial_{x^1 x^1} & \partial_{x^1 x^2} & \cdots \\ \partial_{x^2 x^1} & \partial_{x^2 x^2} & \cdots \\ \vdots & & \ddots \end{pmatrix}$$

$$= \sum_{k=1}^{n} \left(\mu^X_k(t,x) - \sigma^X_{k\cdot}(t,x) \cdot \lambda(t,x) \right) \partial_{x_k} g(t,x)$$

$$+ \frac{1}{2} \sum_{j,k=1}^{n} \Omega^X_{jk}(t,x)\, \partial_{x_j x_k}\, g(t,x)$$

$$\Omega^X(t,x) = \sigma^X(t,x)\, \sigma^X(t,x)'$$

$$g(t,x) = \mathbb{E}^{\mathbb{Q}}_{t,x}\left[e^{-\int_t^T r(u, X_u)\, du}\; G(X_T) \right]$$

$$d X_t = \left(\mu^X(t, X_t) - \sigma^X(t, X_t)\, \lambda(t, X_t) \right) dt$$

$$dX_t = (\mu^X(t,X_t) - \sigma^X(t,X_t)\,\lambda(t,X_t))\,dt + \sigma^X(t,X_t)\,dW_t^{\mathbb{Q}}$$

$W^{\mathbb{Q}} = ((W_t^{\mathbb{Q},1}, \dots, W_t^{\mathbb{Q},n}))_{t\ge 0}$ are $\mathbb{Q}$-independent B. mtn.

$$W^{\mathbb{Q}}_t = +\int_0^t \lambda(u,X_u)\,du + W_t^{\mathbb{P}}$$

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}\left(\int_0^T \lambda(u,X_u)\,dW_u^{\mathbb{P}}\right)$$

no arb $\iff \exists\, \overset{\mathbb{Q}^N}{\mathbb{Q}} \sim \mathbb{P}$ and a numeraire N s.t.

$$\frac{g_t}{\underbrace{B_t}_{N_t}} = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{g_T}{\underbrace{B_T}_{N_T}}\right]$$

for all traded assets g.

an arb is self-financing strategy $(\alpha_t^{(1)}, \alpha_t^{(2)}, \dots, \alpha_t^{(m)}, \beta_t)_{t\ge 0}$ in $(f_t^{(1)}, f_t^{(2)}, \dots, f_t^{(m)}, B_t)$

m maybe different than n.

if $m < n$ markets are said

to be incomplete

s.t. $V_0 = 0$, $\exists\ t_a$ s.t. a) $\mathbb{P}(V_{t_a} \geq 0) = 1$

b) $\mathbb{P}(V_{t_a} > 0) > 0$

$$V_t = \sum_{k=1}^{m} \alpha_t^{(k)} f_t^{(k)} + \beta_t B_t$$

$$dV_t \underset{\substack{\uparrow \\ \text{self-financing}}}{=} \sum_{k=1}^{m} \alpha_t^{(k)} df_t^{(k)} + \beta_t \, dB_t$$

$$\mathbb{E}_t^{\mathbb{Q}}\left[d\left(\frac{V_t}{B_t}\right) \right] = 0 \ ?$$

$dB_t = r_t B_t dt$

$d\left(\frac{1}{B_t}\right) = \frac{-r_t}{B_t} dt$

$$d\left(\frac{V_t}{B_t}\right) = \frac{dV_t}{B_t} + V_t \, d\left(\frac{1}{B_t}\right)$$

$$= \frac{dV_t}{B_t} + V_t \left(\frac{-r_t}{B_t} dt \right)$$

$$d\left(\frac{V_t}{B_t}\right) = \sum_{k=1}^{m} \alpha_t^{(k)} \frac{df_t^{(k)}}{B_t} + \beta_t \frac{dB_t}{B_t} - \frac{V_t r_t}{B_t} dt$$

$$\mathbb{E}_t^{\mathbb{Q}}\left[d\left(\frac{V_t}{B_t}\right) \right] = \sum_{k=1}^{m} \alpha_t^{(k)} \frac{r_t f_t^{(k)}}{B_t} dt + \beta_t r_t dt - \frac{V_t r_t dt}{B_t}$$

$$= \frac{r_t}{B_t} \left\{ \underbrace{\sum \alpha_t^{(k)} f_t^{(k)} + \beta_t B_t}_{} - V_t \right\} dt = 0$$

V_t

$$\mathbb{E}_t^{\mathbb{Q}}\left[d\left(\frac{f_t^{(n)}}{B_t}\right)\right] = 0$$

$$\Rightarrow \mathbb{E}_t^{\mathbb{Q}}\left[\frac{d f_t^{(n)}}{B_t} - \frac{f_t^{(n)} \, r_t}{B_t} \, dt\right] = 0$$

$$\Rightarrow \mathbb{E}_t^{\mathbb{Q}}\left[\frac{d f_t^{(n)}}{B_t}\right] = \frac{f_t^{(n)} \, r_t}{B_t} \, dt$$

① if $\frac{F^{(n)}}{B}$ are all $\mathbb{Q}$-mtg, then

$\frac{V}{B}$ is a $\mathbb{Q}$-mtg for

self-financing strategies.

② assume $\exists$ $\mathbb{Q} \sim \mathbb{P}$ s.t. $\frac{F^{(n)}}{B}$ are $\mathbb{Q}$-mtgs.

also suppose $\exists$ t_a and self-financing strategy (α, β) s.t.

ⓐ $\mathbb{P}(V_{t_a} \geq 0) = 1$ & ⓑ $\mathbb{P}(V_{t_a} > 0) > 0$

$\mathbb{Q} \sim \mathbb{P} \Rightarrow$ $\mathbb{Q}(V_{t_a} \geq 0) = 1$ & $\mathbb{Q}(V_{t_a} > 0) > 0$

$B > 0$ a.s. $\Rightarrow$ $\quad \mathbb{Q}\left(\dfrac{V_{ta}}{B_{ta}} \geq 0 \right) = 1 \quad \& \quad \mathbb{Q}\left(\dfrac{V_{ta}}{B_{ta}} > 0 \right) > 0$

ⓒ $\qquad$ ⓓ

also we know that

$$0 < \mathbb{E}_0^{\mathbb{Q}}\left[\frac{V_{ta}}{B_{ta}} \right] = \frac{V_0}{B_0} \qquad \text{(because of ①)}$$

└ b/c of ⓒ & ⓓ

and hence there can be no arbitrage.

Application: Heston model W^S, W^v are $\mathbb{P}$-Bmtn

$$dS_t = S_t(\mu\, dt + \sqrt{v_t}\, dW_t^S)$$

$$dv_t = \kappa(\theta - v_t)\, dt + \eta\sqrt{v_t}\, dW_t^v$$

$$d[W^S, W^v]_t = \rho\, dt$$

want to find a $\mathbb{Q} \sim \mathbb{P}$ s.t. $\frac{S}{B}$ is a $\mathbb{Q}$-mtg.

$$W_t^S = W_t^1$$

$$W_t^v = \rho W_t^1 + \sqrt{1-\rho^2}\, W_t^2$$

W^1 & W^2 are independent

$$W_t^v \underset{\mathbb{P}}{\sim} \mathcal{N}(0, t)$$

$$W_t^S \underset{\mathbb{P}}{\sim} \mathcal{N}(0, t)$$

$$\mathbb{C}[W_t^S, W_t^v] = \rho t$$

$$\frac{d\mathbb{Q}^{\lambda^1, \lambda^2}}{d\mathbb{P}} = \mathcal{E}\left(\int_0^T \lambda_u^{(1)}\, dW_u^{(1)}\right)\mathcal{E}\left(\int_0^T \lambda_u^{(2)}\, dW_u^{(2)}\right)$$

$$\hookrightarrow e^{-\frac{1}{2}\int_0^T (\lambda_u^{(1)})^2\, du + \int_0^T \lambda_u^{(1)}\, dW_u^{(1)}}$$

NB: λ^1 & λ^2 may both depend on W^1 & W^2 and maybe their history.

Girsanov's Theorem

$$\Rightarrow \qquad W_t^{1*} = -\int_0^t \lambda_u^{(1)}\,du + W_t^{(1)} \qquad \times \rho$$

$$W_t^{2*} = -\int_0^t \lambda_u^{(2)}\,du + W_t^{(2)} \qquad \times \sqrt{1-\rho^2}$$

are independent $\mathbb{Q}^{\lambda^1,\lambda^2}$ B.mtn.

$$W_t^{S} = \overbrace{W_t^{1*}}^{X_t^*} + \int_0^t \lambda_u^{(1)}\,du$$

$$W_t^{v} = \underbrace{\rho W_t^{1*} + \sqrt{1-\rho^2}\,W_t^{2*}}_{Y_t^*} + \rho\int_0^t \lambda_u^{(1)}\,du + \sqrt{1-\rho^2}\int_0^t \lambda_u^{(2)}\,du$$

$$W_t^{S*} = -\int_0^t \overbrace{\lambda_u^{(1)}}^{-\lambda_t^{(S)}}\,du + W_t^{S}$$

$$W_t^{v*} = -\int_0^t \overbrace{\left(\rho\,\lambda_u^{(1)} + \sqrt{1-\rho^2}\,\lambda_u^{(2)}\right)}^{-\lambda_t^{(v)}}\,du + W_t^{v}$$

are correlated $\mathbb{Q}^{\lambda^1,\lambda^2}$ - B mtn. and

$$d[W^S, W^v]_t = \rho\,dt$$

$$dS_t = S_t\left(\mu\,dt + \sqrt{v_t}\left(dW_t^{S*} - \lambda_t^S\,dt\right)\right)$$

$$= S_t\left(\boxed{(\mu - \sqrt{v_t}\,\lambda_t^S)}\,dt + \sqrt{v_t}\,dW_t^{S*}\right)$$

$= r_t$ to make S/B a $\mathbb{Q}$-mtg.

$$dv_t = \kappa(\theta - v_t)\,dt + \eta\sqrt{v_t}\left(dW_t^{v*} - \lambda_t^v\,dt\right)$$

$$= \left[\kappa(\theta - v_t) - \eta\sqrt{v_t}\,\lambda_t^v\right]dt + \eta\sqrt{v_t}\,dW_t^{*v}$$

making S/B a $\mathbb{Q}^{\lambda^1,\lambda^2}$ mtg ⇒

$$\lambda_t^{(1)} = \lambda_t^S = \frac{\mu - r}{\sqrt{v_t}}$$

but λ_t^v is still arbitrary.

a set of parametric choices of λ^v ...

$$\lambda_t^v = a\sqrt{v_t} + \frac{b}{\sqrt{v_t}}$$

drift of v $= \kappa(\theta - v_t) - \eta\sqrt{v_t}\left(a\sqrt{v_t} + \frac{b}{\sqrt{v_t}}\right)$

$$= (\kappa\theta - \eta b) - (\kappa + a\eta)\, v_t$$

$$= (\kappa + a\eta)\left[\frac{\kappa\theta - \eta b}{\kappa + a\eta} - v_t\right]$$

$$= \kappa^*\left[\theta^* - v_t\right]$$

$\mathbb{Q}$ model is

$$dS_t = S_t\left(r\,dt + \sqrt{v_t}\,dW_t^{*S}\right)$$

$$dv_t = \underset{\hookrightarrow \kappa}{\kappa^*}\left(\underset{\hookrightarrow \theta}{\theta^*} - v_t\right)dt + \eta\sqrt{v_t}\,dW_t^{*v}$$

$$g(t,S,v) = \mathbb{E}^{\mathbb{Q}}_{t,S,v}\left[e^{-r\tau}\, G(S_T)\right]$$

$$(\partial_t + \mathcal{L}_t^{S,v})\, g(t,S,v) = r\, g(t,S,v), \quad g(T,S,v) = G(S)$$

$$\mathcal{L}_t^{S,v} = r S \partial_S + \frac{1}{2} S^2 v\, \partial_{SS}$$

$$\mathcal{L}_t^{s,v} = r s \partial_s + \frac{1}{2} s^2 v \partial_{ss}$$

$$+ \kappa(\theta - v)\partial_v + \frac{1}{2}\eta^2 v \partial_{vv} + \eta^{s} v \partial_{sv}$$

this is not a nice PDE... but if we set

$$X_t = \log(S_t) \Rightarrow$$

$$dX_t = \left(0 + r S_t \cdot \frac{1}{S_t} + \frac{1}{2} v_t S_t^2 \left(-\frac{1}{S_t^2}\right)\right)dt$$

$$+ \sqrt{v_t} \cdot S_t \cdot \frac{1}{S_t} \, dW_t^{*S}$$

$$= (r - \tfrac{1}{2} v_t)\,dt + \sqrt{v_t}\, dW_t^{*S}$$

$$h(t,x,v) = \mathbb{E}^{\mathbb{Q}}_{t,x}\left[e^{-r\tau} G(e^{X_T}) \right]$$

$$(\partial_t + \mathcal{L}_t^{x,v})\, h(t,x,v) = r\, h(t,x,v)$$

$$\mathcal{L}_t^{x,v} = (r - \tfrac{1}{2} v)\partial_x + \tfrac{1}{2} v \partial_{xx}$$

$$+ \kappa(\theta - v)\partial_v + \tfrac{1}{2}\eta v \partial_{vv} + \rho\eta v \partial_{xv}$$

PDEs with linear coefficients in state variables are said to be affine and the corresponding processes are called

affine models