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$\mathbb{Q}$ risk-neutral measure

no arb $\iff \exists\, \mathbb{P}^* \sim \mathbb{P}$ s.t.

$$\frac{g_t}{B_t} = \mathbb{E}_t^{\mathbb{P}^*}\left[\frac{g_s}{B_s}\right] \quad \forall s \geq t$$

for all traded assets g.

($B = (B_t)_{t\geq 0}$ is the bank account)

Consider the Radon-Nikodym derivative:

$$\frac{d\mathbb{Q}^N}{d\mathbb{Q}} = \frac{N_T / N_0}{B_T / B_0}$$

$N = (N_t)_{t\geq 0}$ is a numeraire asset, i.e.

($N_t > 0$ a.s. $\forall t$. and N is the price process of a traded asset)

i) clearly $\frac{d\mathbb{Q}^N}{d\mathbb{Q}} > 0$ a.s.

ii) $\mathbb{E}^{\mathbb{Q}}\left[\frac{d\mathbb{Q}^N}{d\mathbb{Q}}\right] = \frac{B_0}{N_0} \cdot \mathbb{E}^{\mathbb{Q}}\left[\frac{N_T}{B_T}\right]$

$= 1$

Recall that: $\quad G \in \mathcal{F}_{T_0}$-measurable $\quad T_0 \leq T$

$$\mathbb{E}^{\mathbb{Q}^N}[G] = \mathbb{E}^{\mathbb{Q}}\left[G \, \frac{d\mathbb{Q}^N}{d\mathbb{Q}}\right]$$

$$\mathbb{E}^{\mathbb{Q}^N}[G \mid \mathcal{F}_t] = \frac{\mathbb{E}^{\mathbb{Q}}\left[G \, \frac{d\mathbb{Q}^N}{d\mathbb{Q}} \,\middle|\, \mathcal{F}_t\right]}{\mathbb{E}^{\mathbb{Q}}\left[\frac{d\mathbb{Q}^N}{d\mathbb{Q}} \,\middle|\, \mathcal{F}_t\right]} \quad t < T_0$$

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{d\mathbb{Q}^N}{d\mathbb{Q}} \,\middle|\, \mathcal{F}_t \right]$$

also recall that: i) $\mathbb{E}^{\mathbb{Q}}\left[\frac{d\mathbb{Q}^N}{d\mathbb{Q}} \right] = 1$

$$\eta_t = \left.\frac{d\mathbb{Q}^N}{d\mathbb{Q}}\right|_{\mathcal{F}_t} = \mathbb{E}^{\mathbb{Q}}\left[\frac{d\mathbb{Q}^N}{d\mathbb{Q}} \,\middle|\, \mathcal{F}_t \right]$$

η is a martingale

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check: $\mathbb{E}^{\mathbb{Q}^N}\left[\frac{g_s}{N_s} \,\middle|\, \mathcal{F}_t \right] \stackrel{?}{=} \frac{g_t}{N_t}$ $\quad s \in [t, T]$

$$\text{lhs} = \frac{\mathbb{E}^{\mathbb{Q}}\left[\frac{g_s}{N_s} \cdot \frac{d\mathbb{Q}^N}{d\mathbb{Q}} \,\middle|\, \mathcal{F}_t \right]}{\mathbb{E}^{\mathbb{Q}}\left[\frac{d\mathbb{Q}^N}{d\mathbb{Q}} \,\middle|\, \mathcal{F}_t \right]}$$

$$= \frac{\mathbb{E}^{\mathbb{Q}}\left[\frac{g_s}{N_s} \cdot \frac{N_T / \cancel{N_0}}{B_T / \cancel{B_0}} \,\middle|\, \mathcal{F}_t \right]}{\underbrace{\mathbb{E}^{\mathbb{Q}}\left[\frac{N_T / \cancel{N_0}}{B_T / \cancel{B_0}} \,\middle|\, \mathcal{F}_t \right]}_{\hookrightarrow\, N_t / B_t}}$$

$$= \frac{B_t}{N_t} \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[\frac{g_s}{N_s} \cdot \frac{N_T}{B_T} \,\middle|\, \mathcal{F}_s \right] \,\middle|\, \mathcal{F}_t \right]$$

$$= \frac{B_t}{N_t} \mathbb{E}^{\mathbb{Q}}\left[\frac{g_s}{\cancel{N_s}} \cdot \frac{\cancel{N_s}}{B_s} \,\middle|\, \mathcal{F}_t \right]$$

$$= \frac{B_t}{N_t}\,\mathbb{E}^{\mathbb{Q}}\left[\frac{g_s}{\cancel{N_s}}\cdot\frac{\cancel{N_s}}{B_s}\,\Big|\,\mathcal{F}_t\right]$$

$$= \frac{B_t}{N_t}\,\mathbb{E}^{\mathbb{Q}}\left[\frac{g_s}{B_s}\,\Big|\,\mathcal{F}_t\right] = \frac{B_t}{N_t}\cdot\frac{g_t}{B_t}$$

$$= \frac{g_t}{N_t} = \text{r.h.s.}$$

since $\dfrac{d\mathbb{Q}^N}{d\mathbb{Q}} > 0$ a.s. we have $\mathbb{Q}^N \sim \mathbb{Q}$

and since $\mathbb{Q} \sim \mathbb{P} \Rightarrow \mathbb{Q}^N \sim \mathbb{P}$

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no arb $\iff \exists\ \mathbb{Q}^N \sim \mathbb{P}$ s.t.

$$\frac{g_t}{N_t} = \mathbb{E}^{\mathbb{Q}^N}\left[\frac{g_s}{N_s}\,\Big|\,\mathcal{F}_t\right] \qquad s \geq t$$

$\forall$ all traded g and numeraire N.

$$\frac{d\mathbb{Q}^F}{d\mathbb{Q}^E} = \frac{d\mathbb{Q}^F}{d\mathbb{Q}} \Big/ \frac{d\mathbb{Q}^E}{d\mathbb{Q}} = \frac{F_T/F_0}{B_T/B_0} \Big/ \frac{E_T/E_0}{B_T/B_0} = \frac{F_T/F_0}{E_T/E_0}$$

$$\begin{tikzcd}
\mathbb{Q}^F \arrow[rr, phantom, "\sim"] & & \mathbb{Q}^E \arrow[ll, bend right] \\
& \mathbb{Q} \arrow[ul, bend left, "\sim"] \arrow[ur, bend right, "\sim"'] & \\
& \mathbb{P} \arrow[u, phantom, "\sim"] &
\end{tikzcd}$$

$$\frac{d\mathbb{Q}^F}{d\mathbb{Q}} = \frac{F_T/F_0}{B_T/B_0} \qquad \frac{d\mathbb{Q}^E}{d\mathbb{Q}} = \frac{E_T/E_0}{B_T/B_0}$$

$$\boxed{\frac{d\mathbb{Q}^F}{d\mathbb{Q}^E} = \frac{F_T / F_0}{E_T / E_0}}$$

is the Radon-Nikodym which connects the two numeraire measures

How are Brownian motions connected under

$\mathbb{Q}^N - \mathbb{Q}$ & $\mathbb{Q}^F - \mathbb{Q}^G$

$$\frac{d\mathbb{Q}^N}{d\mathbb{Q}} = \frac{N_T / N_0}{B_T / B_0}$$

Girsanov's Thm: $\displaystyle \frac{d\mathbb{Q}^*}{d\mathbb{Q}} = \mathcal{E}\left(\int_0^T \eta_s \, dW_s\right)$

W is a $\mathbb{Q}$-B. mtn

$W^* = (W^*_t)$ $\quad W^*_t = -\int_0^T \eta_s \, ds + W_t$

W^* is a $\mathbb{Q}^*$ - B. mtn

$\mathfrak{z} = (\mathfrak{z}_t)_{t \geq 0}$ s.t. $\displaystyle \frac{d\mathfrak{z}_t}{\mathfrak{z}_t} = \eta_t \, dW_t$

$$\mathfrak{z}_T = \mathcal{E}\left(\int_0^T \eta_s \, dW_s\right)$$

$$= \exp\left\{-\frac{1}{2}\int_0^T \eta_s^2 \, ds + \int_0^T \eta_s \, dW_s\right\}$$

$N_t = N(t, X_t)$ due to Markov property of the system. (part of our assumptions)

$N : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}$

$C^{1,2}$

$W^{\mathbb{P}} = (W^{\mathbb{P}}_t)_{t \geq 0}$ a $\mathbb{P}$-B. mtn

$$dN_t = N_t \left(\mu^N(t, X_t)\, dt + \sigma^N(t, X_t)\, dW^{\mathbb{P}}_t\right)$$

$$= N_t \left(r(t, X_t)\, dt + \sigma^N(t, X_t)\, dW^{\mathbb{Q}}_t\right)$$

b/c N is the price of a traded asset!

define $\mathfrak{Z} = (\mathfrak{Z}_t)_{t\geq 0}$

$$\mathfrak{Z}_t = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{N_T/N_0}{B_T/B_0}\right] = \frac{N_t/N_0}{B_t/B_0}$$

$$B_t = e^{+\int_0^t r_s\,ds}$$

in fact $\mathfrak{Z}$ is a martingale

$$d\mathfrak{Z}_t = \Big\{ d(N_t)\frac{1}{B_t} + N_t\, d\left(\frac{1}{B_t}\right) + \underbrace{d[N, \tfrac{1}{B}]_t}_{0} \Big\}\frac{B_0}{N_0}$$

0 since B and hence $\frac{1}{B}$ is differentiable.

NB: $d\left(\frac{1}{B_t}\right) = -r_t \cdot \left(\frac{1}{B_t}\right) dt$

$$\left(Y_t = 1/B_t = e^{-\int_0^t r_s\,ds}, \quad dY_t = -r_t\, e^{-\int_0^t r_s\,ds}\, dt \right)$$

$$\Rightarrow \quad d\mathfrak{Z}_t = \Big\{ N_t(\cancel{r_t}\,dt + \sigma_t^N\, dW_t^{\mathbb{Q}})\cdot\left(\frac{1}{B_t}\right)$$

$$+ N_t\left(-\cancel{r_t \frac{1}{B_t}}\,dt\right) + 0 \Big\}\frac{B_0}{N_0}$$

$$= \underbrace{\left(\frac{N_t/N_0}{B_t/B_0}\right)}_{\mathfrak{Z}_t} \cdot \sigma_t^N\, dW_t^{\mathbb{Q}}$$

$$\Rightarrow \quad d\mathfrak{Z}_t = \mathfrak{Z}_t \cdot \sigma_t^N\, dW_t^{\mathbb{Q}}$$

Girsanov's Theorem:

$\Rightarrow$ $W^{\mathbb{Q}^N} = (W_t^{\mathbb{Q}^N})_{t\geq 0}$ is a $\mathbb{Q}^N$-B.motn where

$$\boxed{W_t^{\mathbb{Q}^N} = -\int_0^t \sigma_s^N\, ds + W_t^{\mathbb{Q}}}$$

Black-Scholes

a) $dS_t = S_t(\mu\,dt + \sigma\,dW_t)$

b) $B_t = e^{rt}$

value a call ... $g_T = (S_T - K)_+$

$$= \underbrace{S_T\,\mathbb{1}_{S_T>K}}_{h_T} - \underbrace{K\,\mathbb{1}_{S_T>K}}_{f_T}$$

$$\frac{f_t}{B_t} = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{f_T}{B_T}\right]$$

$$\Rightarrow f_t = B_t\,\mathbb{E}_t^{\mathbb{Q}}\left[\frac{K\,\mathbb{1}_{S_T>K}}{B_T}\right]$$

$$= e^{-r\overbrace{\tau}^{T-t}}\,K\,\mathbb{Q}_t(S_T>K)$$

$$\frac{h_t}{B_t} = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{h_T}{B_T}\right]$$

$$\Rightarrow h_t = B_t\,\mathbb{E}_t^{\mathbb{Q}}\left[\frac{S_T\,\mathbb{1}_{S_T>K}}{B_T}\right]$$

$$= e^{-r\tau}\,\mathbb{E}_t^{\mathbb{Q}}\left[S_T\,\mathbb{1}_{S_T>K}\right]$$

since $S_t > 0$ a.s. $\forall t$ it can be used

as a numeraire asset

$$\frac{h_t}{S_t} = \mathbb{E}_t^{\mathbb{Q}^S}\left[\frac{h_T}{S_T}\right]$$

$$\Rightarrow \quad h_t = S_t \, \mathbb{E}_t^{\mathbb{Q}^S}\left[\frac{S_T \, \mathbb{1}_{S_T > K}}{S_T} \right]$$

$$= S_t \, \mathbb{E}_t^{\mathbb{Q}^S}\left[\mathbb{1}_{S_T > K} \right]$$

$$= S_t \, \mathbb{Q}_t^S (S_T > K)$$

$$dS_t = S_t (\mu \, dt + \sigma \, dW_t)$$

$$= S_t (r \, dt + \sigma \, dW_t^{\mathbb{Q}})$$

$\Rightarrow$

$$S_T = S_t \, e^{(r - \frac{1}{2}\sigma^2)\tau + \sigma (W_T^{\mathbb{Q}} - W_t^{\mathbb{Q}})}$$

$$\Rightarrow \quad \log (S_T / S_t) \overset{d}{=} (r - \tfrac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau} \, Z$$

$$Z \underset{\mathbb{Q}}{\sim} \mathcal{N}(0,1)$$

$$\mathbb{Q}_t(S_T > K) = \mathbb{Q}_t\left(\log\left(\frac{S_T}{S_t}\right) > \log(K/S_t) \right)$$

$$= \mathbb{Q}_t\left((r - \tfrac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau} \, Z > \log(K/S_t) \right)$$

$$= \mathbb{Q}_t\left(Z > \frac{\log(K/S_t) - (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \right)$$

$$= \mathbb{Q}_t\left(Z < - (\quad\quad) \right)$$

$$= \Phi\left(\frac{\log(S_t/K) + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \right)$$

How about $\mathbb{Q}^S_t(S_T > K)$?

recall: $dS_t = S_t(r\,dt + \sigma\,dW_t^{\mathbb{Q}})$

$W^{\mathbb{Q}^S} = (W_t^{\mathbb{Q}^S})_{t\geq}$, s.t. $W_t^{\mathbb{Q}^S} = -\int_0^t \sigma\,ds + W_t^{\mathbb{Q}}$

is a $\mathbb{Q}^S$-B.mtn.

or $W_t^{\mathbb{Q}^S} + \sigma t = W_t^{\mathbb{Q}}$

$$\Rightarrow dS_t = S_t(r\,dt + \sigma(dW_t^{\mathbb{Q}^S} + \sigma\,dt))$$

$$= S_t((r + \sigma^2)\,dt + \sigma\,dW_t^{\mathbb{Q}^S})$$

$$\Rightarrow S_T = S_t\, e^{((r+\sigma^2) - \frac{1}{2}\sigma^2)\tau + \sigma(W_T^{\mathbb{Q}^S} - W_t^{\mathbb{Q}^S})}$$

$$\Rightarrow \log\left(\frac{S_T}{S_t}\right) \stackrel{d}{=} (r + \tfrac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\, Z^{\mathbb{Q}^S}$$

$$Z^{\mathbb{Q}^S} \underset{\mathbb{Q}^S}{\sim} N(0,1)$$

$$\Rightarrow \mathbb{Q}^S_t(S_T > K) = \Phi\left(\frac{\log(S_t/K) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \right)$$

all together now...

$$g_t = h_t - f_t$$

$$= S_t \, \Phi(d_+) - e^{-r\tau} K \, \Phi(d_-)$$

$$d_\pm = \frac{\log(S_t/K) + (r \pm \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$$

F, G two numeraires

$$dF_t = F_t (r_t\, dt + \sigma_t^F\, dW_t^{\mathbb{Q}})$$
$$dG_t = G_t (r_t\, dt + \sigma_t^G\, dW_t^{\mathbb{Q}})$$

$$W_t^{\mathbb{Q}^F} = - \int_0^t \sigma_s^F\, ds + W_t^{\mathbb{Q}} \quad \text{is a } \mathbb{Q}^F\text{-B.motion}$$
$$W_t^{\mathbb{Q}^G} = - \int_0^t \sigma_s^G\, ds + W_t^{\mathbb{Q}} \quad \text{is a } \mathbb{Q}^G\text{-B.motion}$$

$$\frac{d\mathbb{Q}^G}{d\mathbb{Q}^F} = \frac{G_T / G_0}{F_T / F_0}$$

$$\zeta_t = \mathbb{E}_t^{\mathbb{Q}^F}\left[\frac{G_T / G_0}{F_T / F_0} \right] = \frac{G_t}{F_t} \cdot \frac{F_0}{G_0}$$

$$d\zeta_t = \left\{ dG_t \left(\frac{1}{F_t}\right) + G_t\, d\left(\frac{1}{F_t}\right) + d\left[G, \frac{1}{F}\right]_t \right\} \frac{F_0}{G_0}$$

$$\ell_t = \ell(t, F_t), \qquad \ell(t,F) = 1/F$$

$$d\ell_t = \Big(\underbrace{\partial_t \ell(t,F_t)}_{0} + F_t r_t \cdot \underbrace{\partial_F \ell(t,F_t)}_{-1/F_t^2}$$
$$+ \frac{1}{2} F_t^2 (\sigma_t^F)^2 \cdot \underbrace{\partial_{FF} \ell(t,F_t)}_{+\frac{2}{F_t^3}} \Big)\, dt$$
$$+ F_t \cdot \sigma_t^F \cdot \underbrace{\partial_F \ell(t,F_t)}_{-1/F_t^2}\, dW_t^{\mathbb{Q}}$$
$$= \ell_t \left((-r_t + (\sigma_t^F)^2)\, dt - \sigma_t^F\, dW_t^{\mathbb{Q}} \right)$$

$$d\left[G, \frac{1}{F}\right]_t = -\ell_t\, G_t\, \sigma_t^F \sigma_t^G\, dt$$

$$d\zeta_t = \zeta_t \Big\{ \big(r_t\, dt + \sigma_t^G\, dW_t^{\mathbb{Q}} \big) + \big((-r_t + (\sigma_t^F)^2)\, dt - \sigma_t^F\, dW_t^{\mathbb{Q}} \big)$$
$$- \sigma_t^F \sigma_t^G\, dt \Big\}$$

$$- \sigma_t^F \sigma_t^G \, dt \Big\}$$

$$= \zeta_t \Big\{ \big((\sigma_t^F)^2 - \sigma_t^F \sigma_t^G \big) dt + (\sigma_t^G - \sigma_t^F) \, dW_t^{\mathbb{Q}} \Big\}$$

$$dW_t^{\mathbb{Q}^F} + \sigma_t^F \, dt$$

$$= \zeta_t \cdot (\sigma_t^G - \sigma_t^F) \, dW_t^{\mathbb{Q}^F}$$

Girsanov $\Rightarrow$ $\quad - \int_0^t (\sigma_s^G - \sigma_s^F) ds + W_t^{\mathbb{Q}^F}$

is a $\mathbb{Q}^G$- B. mtn

Suppose $\zeta_t = \dfrac{X_t}{Y_t}$ and you know ζ is a $\tilde{\mathbb{P}}$-mtg. and $\tilde{\mathbb{P}} \sim \mathbb{P}$

$\Rightarrow \quad d\zeta_t = \zeta_t \cdot$ (vol of X − vol of Y)

written as $\tilde{\mathbb{P}}$ - B-mtn

$$dX_t = X_t (\mu_t^X \, dt + \sigma_t^X \, dW_t^{\mathbb{P}})$$

$$dY_t = Y_t (\mu_t^Y \, dt + \sigma_t^Y \, dW_t^{\mathbb{P}})$$

$$d\zeta_t = \zeta_t (\sigma_t^X - \sigma_t^Y) \, dW_t^{\tilde{\mathbb{P}}}$$