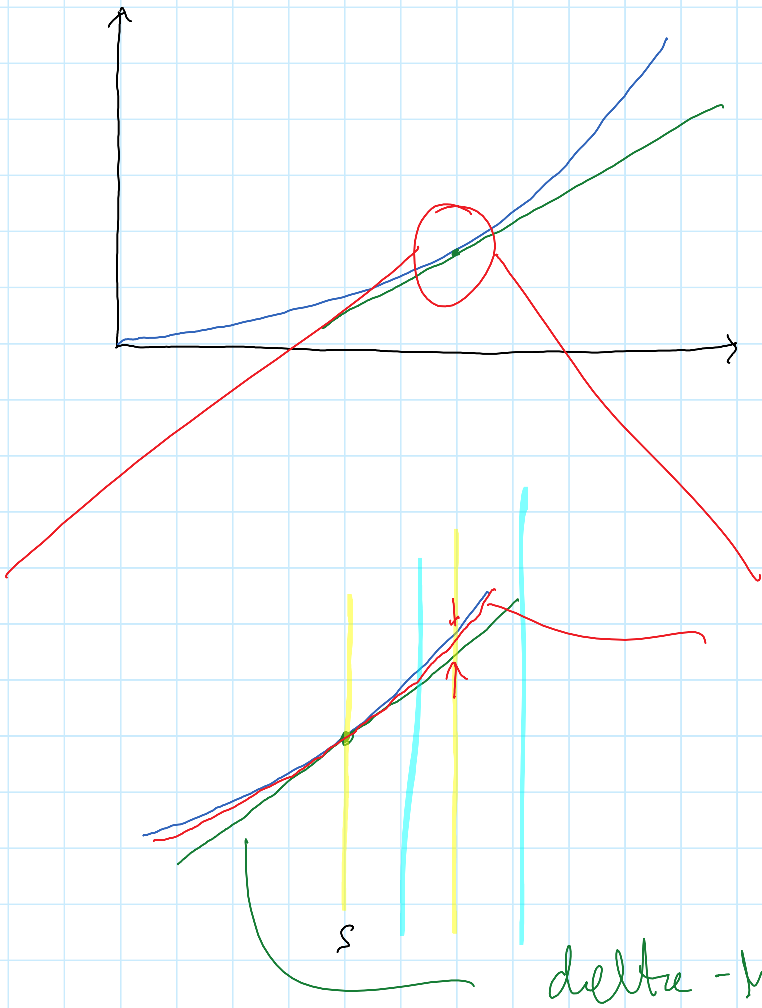


Delta-Gamma hedge ~ quadratic approximation

delta-hedging ~ linear approximation

$$g(t, S + \Delta S)$$

$$= g(t,S) + \Delta S \cdot \overbrace{\partial_S g(t,S)}^{\Delta^g(t,S)}$$

$$+ \tfrac{1}{2}(\Delta S)^2 \underbrace{\partial_{SS} g(t,S)}_{\Gamma^g(t,S)} + \cdots$$

option gamma

now consider a portfolio: another option now consider a portfolio:

another option

$$(\alpha, \beta, \gamma, -1) \text{ in } (S, B, h, g)$$

$$V_t = \alpha_t \overset{f(t,S_t)}{S_t} + \beta_t B_t + \gamma_t \overset{h(t,S_t)}{h_t} - \overset{g(t,S_t)}{g_t}$$

$$\Delta^V = \alpha_t + 0 + \gamma_t \Delta^h_t - \Delta^g_t = 0$$

$$\Gamma^V = 0 + 0 + \gamma_t \Gamma^h_t - \Gamma^g_t = 0$$

$$\Rightarrow \boxed{\gamma_t = \frac{\Gamma^g_t}{\Gamma^h_t}} = \frac{\Gamma^g(t, S_t)}{\Gamma^h(t, S_t)}$$

$$\boxed{\alpha_t = \Delta^g_t - \frac{\Gamma^g_t}{\Gamma^h_t} \cdot \Delta^h_t}$$

t_k: α_{t_k}, M_{t_k}, γ_{t_k} -1

t_{k+1}: $M_{t_k} \longrightarrow$
$$\boxed{\begin{aligned} M_{t_{k+1}} = M_{t_k} e^{r\Delta t} &- (\alpha_{t_{k+1}} - \alpha_{t_k}) S_{t_{k+1}} \\ &- (\gamma_{t_{k+1}} - \gamma_{t_k}) h_{t_{k+1}} \end{aligned}}$$

$$S_{t_k} \longrightarrow S_{t_{k+1}}$$

$$\alpha_{t_k} \longrightarrow \alpha_{t_{k+1}^-} = \alpha_{t_k} = \alpha(t_k, S_{t_k})$$

$$\alpha_{t_k} \longrightarrow \alpha_{t_{k+1}^-} = \alpha_{t_k} = \alpha(t_k, S_{t_k})$$

$$\hookrightarrow \boxed{\alpha_{t_{k+1}} = \alpha(t_{k+1}, S_{t_{k+1}})}$$

$$\gamma_{t_k} \longrightarrow \gamma_{t_{k+1}^-} = \gamma_{t_k} = \gamma(t_k, S_{t_k})$$

$$\hookrightarrow \boxed{\gamma_{t_{k+1}} = \gamma(t_{k+1}, S_{t_{k+1}})}$$

$$h_{t_k} \longrightarrow h_{t_{k+1}} = h(t_{k+1}, S_{t_{k+1}})$$

$$\Gamma(t,S) = \partial_{SS}\, g(t,S)$$
$$= \partial_S\, \Delta(t,S)$$

cdf of a std. normal

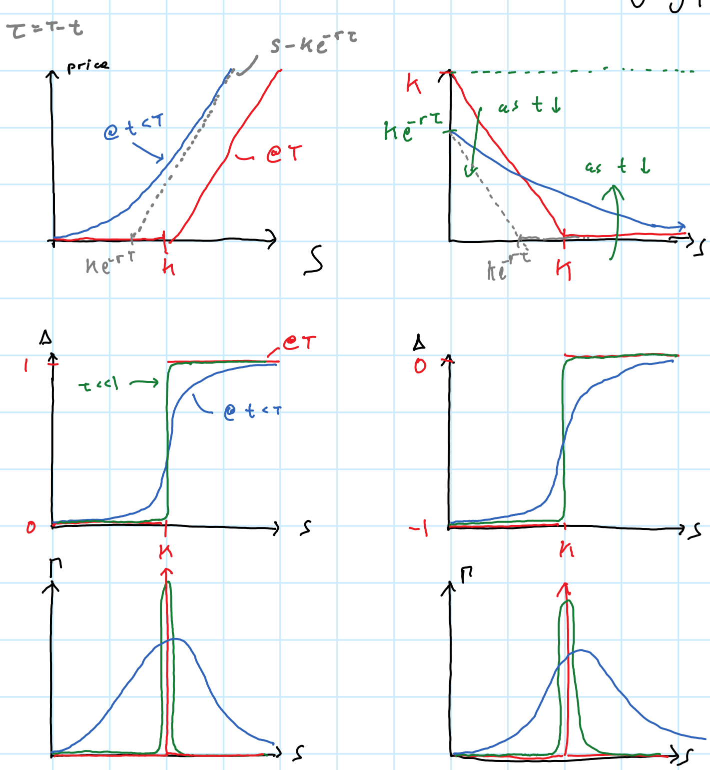
recall that $\Delta^{call}(t,S) = \Phi(d_+)$

$$d_\pm = \frac{\log(S/K) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

pdf of a std. normal

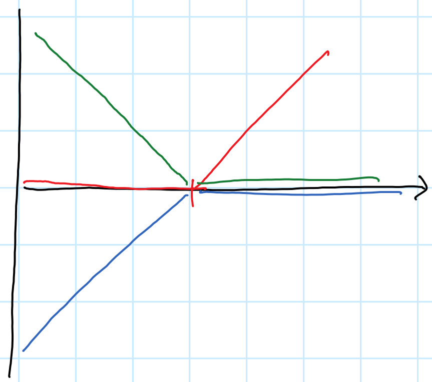
$$\Rightarrow \quad \Gamma^{call}(t,S) = \phi(d_+)\ \partial_S d_+$$
$$= \frac{1}{S\sigma\sqrt{T-t}}\ \phi(d_+)$$

$\tau = T-t$



$$g_T^{put} = (K - S_T)_+ \quad , \quad g_T^{call} = (S_T - K)_+$$

$$g_T^{call} - g_T^{put} = S_T - K$$



$$\Rightarrow \quad g_t^{call} - g_t^{put} = \mathbb{E}_t^{\mathbb{P}^*}\left[e^{-r\tau}(S_T - K) \right]$$

$$= e^{-r\tau}\,\mathbb{E}_t^{\mathbb{P}^*}[S_T] - e^{-r\tau} K$$

$$\boxed{g_t^{call} - g_t^{put} = S_t - e^{-r\tau} K}$$

put - call parity

$$g^{put}(t,S) = K e^{-r\tau}\,\Phi(-d_-) - S\,\Phi(-d_+)$$

$$d_\pm = \frac{\log(S/K) + (r \pm \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$$

$$K - S_T \le g_T^{put} \le K$$

$$\mathbb{E}_t^{\mathbb{P}^*}[K - S_T]\,e^{-r\tau} \le g_t^{put} \le K e^{-r\tau}$$

$$K e^{-r\tau} - S_t \le g_t^{put} \le K e^{-r\tau}$$

$Ke^{...} - S_t \le g'_t \le Ke$

from put-call parity:

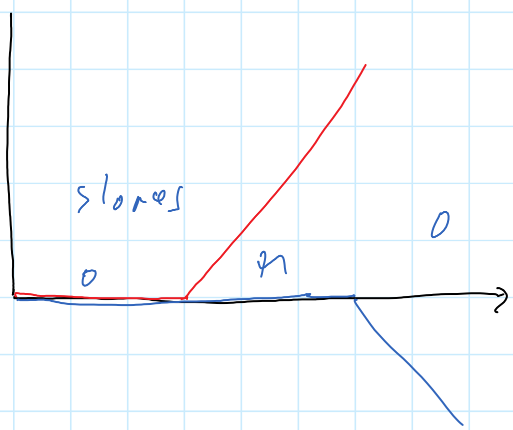
$$g^{call}(t,S) - g^{put}(t,S) = S - Ke^{-r\tau}$$

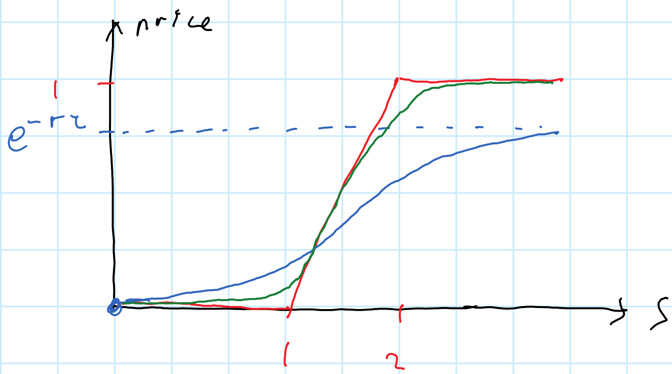
$$\Rightarrow \quad \Delta^{call}(t,S) - \Delta^{put}(t,S) = 1 - 0$$

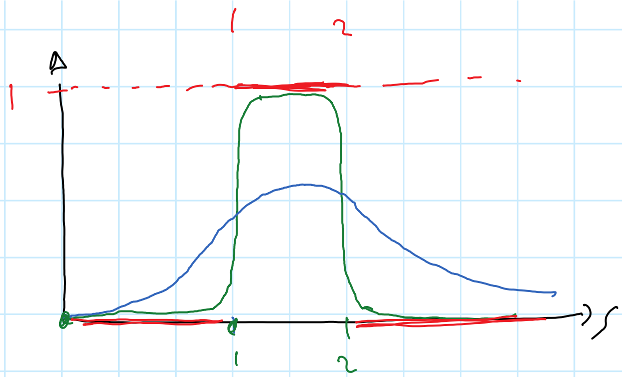
$$\Rightarrow \quad \boxed{\begin{aligned}\Delta^{put}(t,S) &= \Delta^{call}(t,S) - 1 \\ &= \Phi(-d_+)\end{aligned}}$$

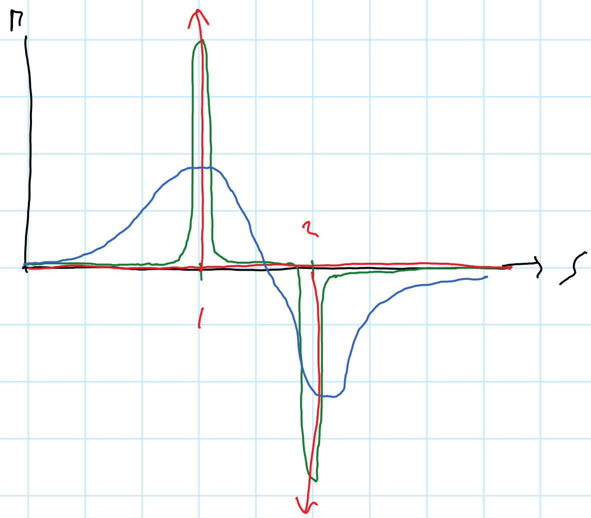
$$\Rightarrow \quad \boxed{\Gamma^{call}(t,S) = \Gamma^{put}(t,S)}$$

long call $K_1 = 1$ & short call $K_2 = 2$






1
2
1
1
2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1

Friday, October 28, 2016 3:18 PM

implied volatility:

$$dS_t = S_t(\mu\,dt + \sigma\,dW_t)$$

$$= S_t(r\,dt + \sigma\,dW_t^*)$$

r_t deterministic function

$$\log\left(\frac{S_T}{S_0}\right) \overset{d}{=} \int_0^T \left(r_u - \frac{1}{2}\sigma^2\right) du + \sigma\sqrt{\tau}\, Z$$

$\tau \triangleq T - t$

$$Z \underset{\mathbb{P}^*}{\sim} \mathcal{N}(0,1)$$

$$g^{call}(t,S;K,T) = S\,\Phi(d_+) - K e^{-\int_t^T r_u du}\,\Phi(d_-)$$

$$d_\pm = \frac{\log(S/K) + \int_t^T (r_u \pm \frac{1}{2}\sigma^2)\,du}{\sigma\sqrt{T-t}}$$

$$g^{call}_{Mkt}(t, S_t; K, T)$$

$\ni \sigma$?

$$(K_1,T_1),\ (K_2,T_2),\ \ldots\ (K_N, T_N)$$

$$\sigma = \underset{\sigma}{\arg\min} \sum_{n=1}^{N} \left(g^{Mkt}(0, S_0; K_n, T_n) \right.$$

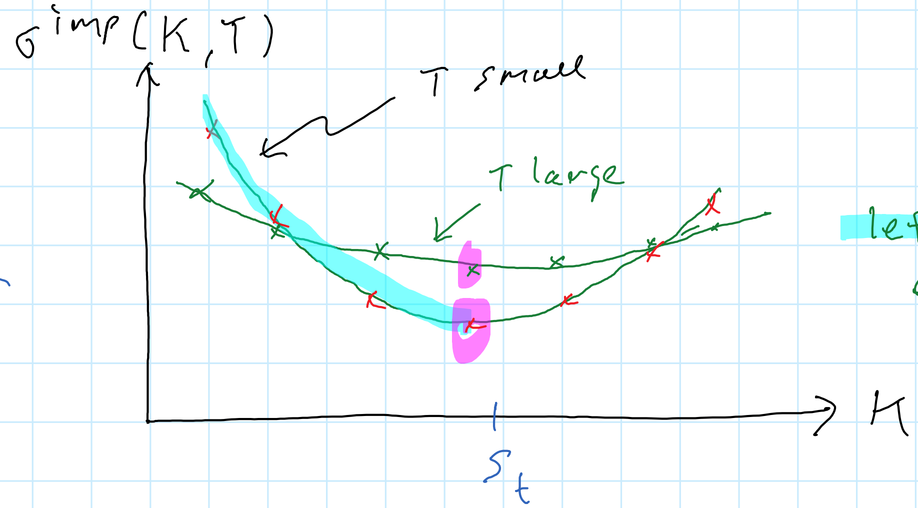
$$\sigma = \arg\min_{\sigma} \sum_{n=1} \Big(g^{Mkt}(0, S_0; K_n, T_n) - g^{Mdl}(0, S_0; K_n, T_n, \sigma) \Big)^2$$

This is not done!

$$\sigma^{imp}(K_j, T_j) = \arg\min_{\sigma} \Big(g^{Mkt}(0, S_0; K_j, T_j) - g^{Mdl}(0, S_0; K_j, T_j, \sigma) \Big)^2$$

implied volatility

smile or smirk

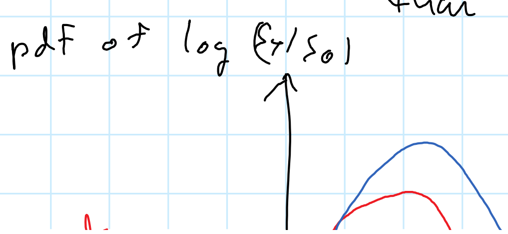


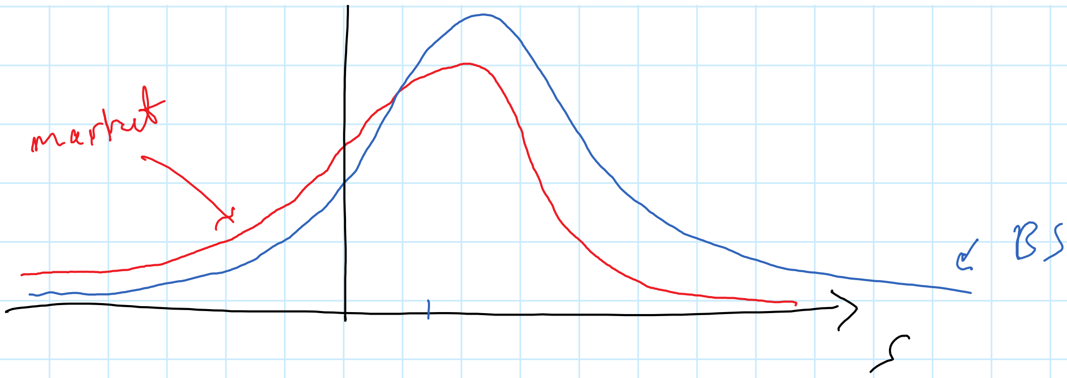
fix T_j

left-skew in equity implied volatilites

Black-Scholes model of GBM asset prices is empirically wrong.

left-skewness $\Rightarrow$ $\log(S_T/S_0)$ has more prob. weight in the left than the right





$\sim S = K$

At-the-money implied volatility can be matched by making

$\sigma \longrightarrow \sigma(t)$ deterministic
piecewise const.