\[ g(t, s + \Delta s) \]

\[ = g(t, s) + \Delta s \cdot \partial_s g(t, s) \]

\[ + \frac{1}{2} (\Delta s)^2 \partial s g(t, s) + \cdots \]

\[ \Delta^2 g(t, s) \]

Delta-Gamma

Hedge 

quadratic

approximation

Delta-hedging 

Linear approximation

Now consider a portfolio:

another option
Now consider a portfolio: 

\[
\begin{bmatrix}
\lambda_t, \beta_t, \gamma_t, -1
\end{bmatrix} \text{ in } \begin{bmatrix}
S_t, B_t, h_t, g(t, s)
\end{bmatrix}
\]

\[V_t = \lambda_t S_t + \beta_t B_t + \gamma_t h_t - g_t\]

\[\Delta_t = \alpha_t + \beta_t \Delta_t - \gamma_t \Delta_t = 0\]

\[\Pi_t = \beta_t + \gamma_t \Pi_t - \gamma_t \Pi_t = 0\]

\[
\Rightarrow \quad \gamma_t = -\frac{\Pi_t g_t}{\Pi_t h_t} = -\frac{\Pi_t g(t, s)}{\Pi_t h(t, s)}
\]

\[\lambda_t = \Delta_t - \frac{\Pi_t g_t}{\Pi_t h_t} \cdot \Delta_t\]

\[
t_{k+1}: \begin{bmatrix}
\alpha_{t_{k+1}}, M_{t_{k+1}}, \gamma_{t_{k+1}}
\end{bmatrix}
\]

\[t_{k+1}: \quad M_{t_{k+1}} = M_{t_k} e^{-\delta t} - (\alpha_{t_{k+1}} - \alpha_{t_k}) S_{t_{k+1}} - (\gamma_{t_{k+1}} - \gamma_{t_k}) h_{t_{k+1}}\]

\[S_{t_{k+1}} = S_{t_k}\]

\[\lambda_{t_{k+1}} = \lambda_{t_k} = \lambda(t_k, S_{t_k})\]
\[
\alpha_{t_n} \rightarrow \alpha_{t_{n+1}} = \alpha_{t_n} = \alpha(t_n, S_{t_n}) \\
\Rightarrow \alpha_{t_{n+1}} = \alpha(t_{n+1}, S_{t_{n+1}})
\]

\[
\gamma_{t_n} \rightarrow \gamma_{t_{n+1}} = \gamma_{t_n} = \gamma(t_n, S_{t_n}) \\
\Rightarrow \gamma_{t_{n+1}} = \gamma(t_{n+1}, S_{t_{n+1}})
\]

\[
h_{t_n} \rightarrow h_{t_{n+1}} = h(t_{n+1}, S_{t_{n+1}})
\]
\[ \Gamma(t,s) = \varphi_s \varphi(t,s) \]

\[ = \varphi_s \Phi(t,s) \]

Recall that \( \Phi(T^u(t,s)) = \Phi(d_t) \)

\[ d_t = \frac{\log(S/K) + \left( \sigma \sqrt{\frac{t}{T-t}} \right) (T-t)}{\sigma \sqrt{T-t}} \]

\[ \Rightarrow \Phi(d_t) = \frac{1}{S \sigma \sqrt{T-t}} \phi(d_t) \]

\[ \phi(d_t) \]
\[
g_{T}^{\text{put}} = (K - S_{T})_{+}, \quad g_{T}^{\text{call}} = (S_{T} - K)_{+}
\]

\[
g_{T}^{\text{call}} - g_{T}^{\text{put}} = S_{T} - K
\]

\[
\Rightarrow g_{T}^{\text{call}} - g_{T}^{\text{put}} = \mathbb{E}_{t}^{\mathbb{P}^{*}} \left[ e^{-rC} (S_{T} - K) \right] = e^{-rC} \mathbb{E}_{t}^{\mathbb{P}^{*}} [S_{T}] - e^{-rC} K
\]

\[
g_{T}^{\text{call}} - g_{T}^{\text{put}} = S_{t} - e^{-rC} K
\]

\[
g_{t}^{\text{call}} = \mathbb{E}_{t}^{\mathbb{P}^{*}} \left[ S_{t} e^{-rC} \right] - \mathbb{E}_{t}^{\mathbb{P}^{*}} \left[ S_{t} e^{-rC} K \right]
\]

\[
g_{t}^{\text{call}}(t, s) = k e^{-rC} \Phi(-d_{-}) - S \Phi(-d_{+})
\]

\[
d_{\pm} = \frac{\log(S/K) + (r + \sigma^{2}/2) T}{\sigma \sqrt{T}}
\]

\[
k - S_{T} \leq g_{T}^{\text{call}} \leq k
\]

\[
\mathbb{E}_{t}^{\mathbb{P}^{*}} [K - S_{T}] e^{-rC} \leq g_{T}^{\text{put}} \leq k e^{-rC}
\]

\[
k e^{-rC} - S_{T} \leq g_{T}^{\text{put}} \leq k e^{-rC}
\]
\[ K - S_t \leq K e^{-r_t} \leq K e^{r_t} \]

From put-call parity:

\[ g^\text{call}(t, s) - g^\text{put}(t, s) = S - K e^{-r_t} \]

\[ \Rightarrow \Delta^\text{call}(t, s) - \Delta^\text{put}(t, s) = 1 - 0 \]

\[ \Rightarrow \quad \Delta^\text{put}(t, s) = \Delta^\text{call}(t, s) - 1 \]

\[ = \Phi(-d_t) \]

\[ \Rightarrow \quad \Delta^\text{call}(t, s) = \Delta^\text{put}(t, s) \]

Long call \( K_1 = 1 \) & short call \( K_2 = 2 \)
implied volatility:

\[ d s_t = s_t \left( \mu dt + \sigma d W_t \right) \]

\[ = s_t \left( \mu dt + \sigma d W_t \right) \]

\[ \text{deterministic function} \]

\[ \log \left( \frac{s_t}{s_0} \right) = \int_0^T \left( r_u - \frac{1}{2} \sigma^2 \right) du + \sigma \sqrt{T-t} \, Z \]

\[ Z \sim N(0,1) \]

\[ g_{\text{cau}}(t,s;K,T) = s_0 \Phi(d_+) - K e^{-\int_0^T r_u du} \Phi(d_-) \]

\[ d_+ = \log \left( \frac{s(t)}{K} \right) + \frac{\int_0^T (r_u - \frac{1}{2} \sigma^2) du}{\sigma \sqrt{T-t}} \]

\[ g_{\text{cau}}(t,s_t;K,T) \]

\[ \left( K_1, T_1 \right), \left( K_2, T_2 \right), \ldots, \left( K_M, T_M \right) \]

\[ \sigma = \arg \min_{\theta} \sum_{i=1}^N \left( g_{\text{cau}}(\theta, S_0; K_i, T_i) \right) \]
\[ \sigma = \arg \min_{\sigma} \sum_{n=1}^{\infty} \left( g^{\text{min}}(0, S_0; K_n, T_n) - g^{\text{ML}}(0, S_0; K_n, T_n, \sigma) \right)^2 \]

This is not done!

\[ \sigma^{\text{imp}}(K_j, T_j) = \arg \min_{\sigma} \left( g^{\text{min}}(0, S_0; K_j, T_j) - g^{\text{ML}}(0, S_0; K_j, T_j, \sigma) \right)^2 \]

Implied volatility

\[ \sigma^{\text{imp}}(K, T) \]

T small

T large

Smile or smirk

Black-Scholes model of CTV asset prices is empirically wrong.

Left-skewness \( \Rightarrow \log(S_T/S_0) \) has more pdf weight in the left than the right.

\[ \text{pdf of } \log(S_T/S_0) \]
At at-the-money implied volatility can be matched by making

\[ \sigma \rightarrow \sigma(t) \text{ deterministic piecewise const.} \]