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- $X = (X_t)_{t \geq 0}$

$$dX_t = \mu^X(t, X_t)\, dt + \sigma^X(t, X_t)\, dW_t$$

- $B = (B_t)_{t \geq 0}$

$$dB_t = r(t, X_t)\, B_t\, dt$$

- $F = (F_t)_{t \geq 0}$

$$dF_t = F_t\,(\mu^F_t\, dt + \sigma^F_t\, dW_t)$$

- value a claim $g = (g_t)_{t \geq 0}$ which pays $g_T = G(X_T)$

$$\frac{\mu^F_t - r_t}{\sigma^F_t} = \frac{\mu^g_t - r_t}{\sigma^g_t} = \underbrace{\lambda^X_t}_{\text{market price of risk}} = \lambda^X(t, X_t)$$

$$g_t = g(t, X_t) \quad , \quad g : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}, \quad g \in C^{1,2}$$

$$dg_t = g_t\,(\mu^g_t\, dt + \sigma^g_t\, dW_t)$$

$$\begin{cases} \partial_t g(t,x) + \left(\mu^X(t,x) - \lambda^X(t,x)\,\sigma^X(t,x)\right)\partial_x g(t,x) \\ \quad + \frac{1}{2}\left(\sigma^X(t,x)\right)^2 \partial_{xx} g(t,x) \end{cases}$$

$$\begin{cases} \partial_t g(t,x) + \left(\mu^x(t,x) - \lambda^x(t,x)\,\sigma^x(t,x)\right)\partial_x g(t,x) \\ \qquad + \frac{1}{2}\left(\sigma^x(t,x)\right)^2 \partial_{xx} g(t,x) \\ = r(t,x)\, g(t,x) \\ g(T,x) = G_T(x) \end{cases}$$

generalized pricing equation

$$g(t,x) = \mathbb{E}^{\mathbb{P}^*}_{t,x}\left[e^{-\int_t^T r(u,X_u)\,du}\, G_T(X_T) \right]$$

where,

$$dX_t = \left(\mu^x(t,X_t) - \lambda^x(t,X_t)\,\sigma^x(t,X_t)\right)dt + \sigma^x(t,X_t)\, dW^*_t$$

$W^* = (W^*_t)_{t\geq 0}$ is a $\mathbb{P}^*$- Brownian

$$W^*_t = \int_0^t \lambda^x(u,X_u)\,du + W_t$$

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \exp\left\{ -\frac{1}{2}\int_0^T \left(\lambda^x(u,X_u)\right)^2 du - \int_0^T \lambda^x(u,X_u)\, dW_u \right\}$$

is the Radon-Nikodym derivative that connects $\mathbb{P}$ and $\mathbb{P}^*$

$$\frac{g_t}{B_t} = \mathbb{E}^{\mathbb{P}^*}\left[\frac{g_T}{B_T} \middle| \mathcal{F}_t \right]$$

no arbitrage $\Longleftrightarrow$ $\exists$ $\mathbb{P}^* \sim \mathbb{P}$ s.t.

$$\tilde{g}_t = \mathbb{E}^{\mathbb{P}^*}\left[\tilde{g}_s \mid \mathcal{F}_t \right] \qquad s,t \in [0,T]$$

$$\tilde{g} = (\tilde{g}_t)_{t \geq 0}, \quad \tilde{g}_t = g_t / B_t.$$

$$W_t^\alpha = W_t + \int_0^t a_u \, du$$

$$\frac{d\mathbb{P}^\alpha}{d\mathbb{P}} = \exp\left\{ -\frac{1}{2}\int_0^T a_u^2 \, du - \int_0^T a_u \, dW_u \right\}$$

what is the $\mathbb{P}^\alpha$- distribution $W_t^\alpha \overset{?}{\underset{\mathbb{P}^\alpha}{\sim}} \mathcal{N}(0,t)$

$$W_s^\alpha - W_t^\alpha \overset{?}{\underset{\mathbb{P}^\alpha}{\sim}} \mathcal{N}(0, s-t)$$

$$W_s^\alpha - W_t^\alpha \overset{?}{\perp} W_v^\alpha - W_u^\alpha \qquad (s,t) \cap (u,v) = \emptyset.$$

$$\mathbb{E}^{\mathbb{P}^\alpha}\left[e^{i\alpha W_t^\alpha} \right]$$

$$= \mathbb{E}^{\mathbb{P}}\left[e^{i\alpha W_t^\alpha} \frac{d\mathbb{P}^\alpha}{d\mathbb{P}} \right] \qquad g_t = e^{-\frac{1}{2}\int_0^t a_u^2 \, du - \int_0^t a_u \, dW_u}$$

$$= \mathbb{E}^{\mathbb{P}}\left[e^{i\alpha W_t^\alpha} g_T \right]$$

$$= \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[e^{i\alpha W_t^\alpha} g_T \,\middle|\, \mathcal{F}_t \right] \right]$$

$$= \mathbb{E}^{\mathbb{P}}\left[e^{i\alpha W_t^\alpha} \underbrace{\mathbb{E}^{\mathbb{P}}\left[g_T \mid \mathcal{F}_t \right]}_{g_t} \right]$$

$$= \mathbb{E}^{\mathbb{P}}\left[e^{i\alpha W_t^\alpha} e^{-\frac{1}{2}\int_0^t a_u^2 \, du - \int_0^t a_u \, dW_u} \right]$$

$$\int_0^t dW_u^\alpha$$

$$= \mathbb{E}^{\mathbb{P}}\left[\underbrace{e^{-\frac{1}{2}\int_0^t a_u^2 \, du - \int_0^t \overbrace{(a_u - i\alpha)}^{\zeta_u} dW_u + i\alpha\int_0^t a_u \, du}}_{A_t} \right]$$

$$\frac{1}{2}\int_0^t \zeta_u^2 - \int_0^t \zeta_u \, dW_u$$

$$h_t = e^{-\frac{1}{2}\int_0^t b_u^2 - \int_0^t b_u \, dW_u} \quad A_t$$

is also a $\mathbb{P}$-mtg

$$\Rightarrow A_t = h_t \cdot e^{+\frac{1}{2}\left(\int_0^t b_u^2 du - \int_0^t a_u^2 du\right) + i\alpha \int_0^t a_u du}$$

$$\hookrightarrow \; a_u^2 - 2i\alpha a_u - \alpha^2$$

$$= h_t \, e^{-\frac{1}{2}\alpha^2 t}$$

$$\Rightarrow \mathbb{E}^{\mathbb{P}^*}\left[e^{i\alpha W_t^*} \right] = e^{-\alpha^2 t} \, \mathbb{E}^{\mathbb{P}}[h_t]$$

$$= e^{-\frac{1}{2}\alpha^2 t} \, h_0$$

$$= e^{-\frac{1}{2}\alpha^2 t}$$

$$W_t^* \underset{\mathbb{P}^*}{\sim} \mathcal{N}(0, t)$$

Dolean-Dade exponential

$$\eta_\cdot = \mathcal{E}\left(\int_0^\cdot a_u \, dW_u \right)$$

$$\Longleftrightarrow d\eta_t = \eta_t \, a_t \, dW_t, \quad \eta_0 = 1$$

$$\Longleftrightarrow \eta_t = \exp\left\{ -\frac{1}{2}\int_0^t a_u^2 \, du + \int_0^t a_u \, dW_u \right\}$$

<u>Girsanov's Theorem:</u>

if $$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \mathcal{E}\left(\int_0^T a_u \, dW_u \right)$$

then

$$W_t^* = -\int_0^t a_u \, du + W_t$$

is a $\mathbb{P}^*$ - B. mtn.

value a call option under Black-Scholes model

- $dS_t = S_t(\mu\, dt + \sigma\, dW_t)$ is traded.
- $dB_t = r B_t\, dt$ $\quad r = \text{const.}$
- $g_T = (S_T - K)_+$

$$\lambda = \frac{\mu - r}{\sigma} = \frac{\mu_t^g - r}{\sigma_t^g}$$

$$\begin{cases} \partial_t g(t,S) + \underbrace{(\mu S - \lambda \cdot \sigma S)}_{\to\, rS} \partial_S g(t,S) + \frac{1}{2}\sigma^2 S^2 \partial_{SS} g(t,S) = r\, g(t,S) \\ g(T,S) = (S-K)_+ \end{cases}$$

$$g(t,S) = \mathbb{E}^{\mathbb{P}^*}_{t,S}\left[e^{-\int_t^T r\, du} (S_T - K)_+ \right]$$

$$dS_t = (\mu S_t - \lambda \cdot \sigma S_t)\, dt + \sigma S_t\, dW_t^*$$

$$= r S_t\, dt + \sigma S_t\, dW_t^*$$

$$W_t^* = \int_0^t \lambda \cdot du + W_t = \lambda \cdot t + W_t$$

is a $\mathbb{P}^*$- B. mtn.

to find $\mathbb{E}^{\mathbb{P}^*}_{t,S}[(S_T - K)_+]$ need $S_T \,|\, \tilde{\mathbb{P}}^*$?

to find $\mathbb{E}^{\mathbb{P}}_{t,S}[\ (S_T - K)_+\ ]$ need $S_T \Big|_{S_t = S} \overset{\mathbb{P}^*}{\sim}$?

$$dS_t = S_t\, r\, dt + S_t\, \sigma\, dW^*_t$$

$$\frac{dS_t}{S_t} = r\, dt + \sigma\, dW^*_t$$

try $\quad F_t = f(t, S_t)$, $\quad f(t, S) = \log(S)$

$$df_t = \Big(\underbrace{\partial_t f(t, S_t)}_{\to 0} + r\, S_t\, \overset{\to 1/S_t}{\partial_S f(t, S_t)} + \tfrac{1}{2}\sigma^2 S_t^2\, \underbrace{\partial_{SS} f(t, S_t)}_{\to -\frac{1}{S_t^2}} \Big)\, dt$$

$$+ \overset{\to 1/S_t}{\partial_S f(t, S_t)}\, \sigma \cdot S_t \cdot dW^*_t$$

$$= \partial_t f(t, S_t)\, dt + \partial_S f(t, S_t)\, dS_t$$

$$+ \tfrac{1}{2}\, \partial_{SS} f(t, S_t)\, (\underbrace{dS_t}_{})^2 + \ldots$$

$$\hookrightarrow \sigma^2 S_t^2\, \underbrace{dW^{*2}_t}_{\hookrightarrow dt}$$

$$\Rightarrow\ df_t = (r - \tfrac{1}{2}\sigma^2)\, dt + \sigma\, dW^*_t$$

$$\Rightarrow\ f_T - f_t = (r - \tfrac{1}{2}\sigma^2)(T - t) + \sigma(W^*_T - W^*_t)$$

$$\Rightarrow\ \log(S_T / S_t) = (r - \tfrac{1}{2}\sigma^2)(T - t) + \sigma(W^*_T - W^*_t)$$

$$\boxed{S_T = S_t \, e^{(r-\frac{1}{2}\sigma^2)(T-t) + \sigma(W^*_T - W^*_t)}}$$

$$\log\left(\frac{S_T}{S_t}\right)\Bigg|_{S_t = s} \underset{\mathbb{P}^*}{\sim} \mathcal{N}\left((r - \tfrac{1}{2}\sigma^2)(T-t) \, ; \, \sigma^2 (T-t) \right)$$

$$H = \mathbb{E}^{\mathbb{P}^*}_{t,s}\left[(S_T - K)_+ \right] \ \ldots$$

$$\log\left(\frac{S_T}{S_t}\right) \overset{d}{=} (r - \tfrac{1}{2}\sigma^2)(T-t) + \sigma\sqrt{T-t}\, Z$$

$$Z \underset{\mathbb{P}^*}{\sim} \mathcal{N}(0,1)$$

$$H = \int_{-\infty}^{\infty} \left(s\, e^{(r-\frac{1}{2}\sigma^2)(T-t) + \sigma\sqrt{T-t}\, z} - K \right)_+ \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}} \, dz$$

$\vdots$ standard calc.

$$= e^{r(T-t)} s\, \Phi(d_+) - K\, \Phi(d_-)$$

$$d_\pm = \frac{\log(s/K) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

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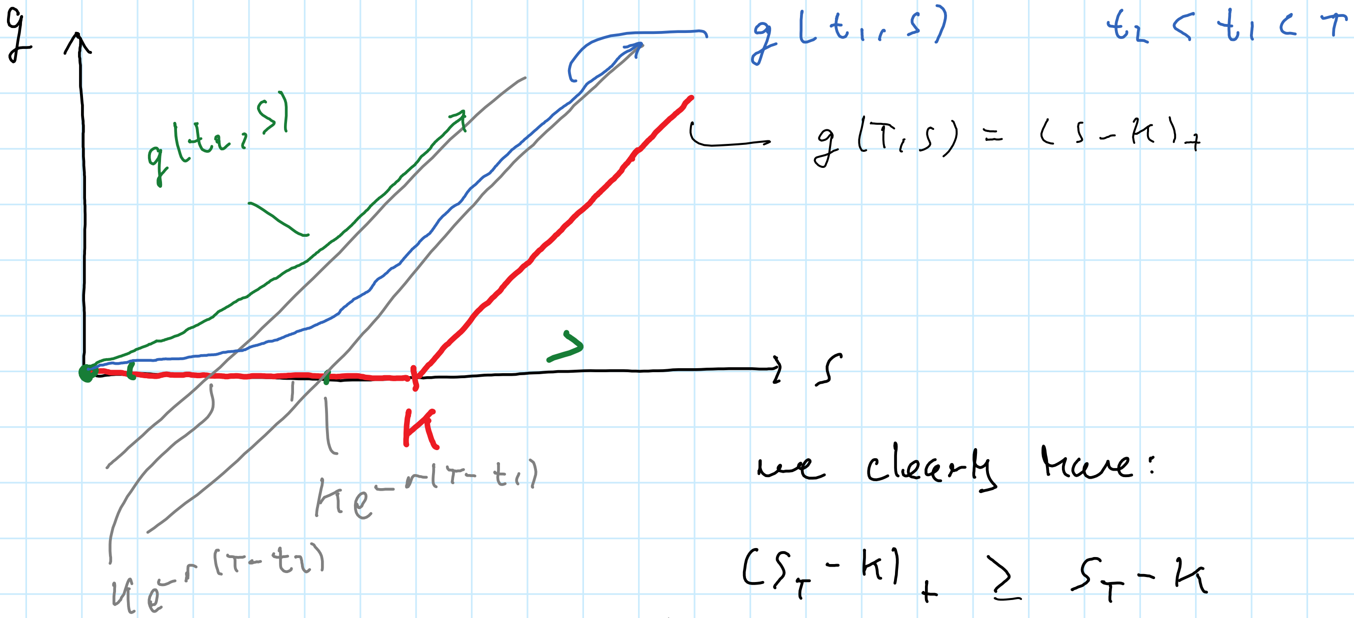
$$g(t,S) = S\,\Phi(d_+) - K e^{-r(T-t)}\,\Phi(d_-)$$

$$d_\pm = \frac{\log(S/K) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$



we clearly have:

$$(S_T - K)_+ \geq S_T - K$$

$$\Rightarrow e^{-r(T-t)}\,\mathbb{E}^{\mathbb{P}^*}_{t,S}\left[(S_T-K)_+\right] \geq e^{-r(T-t)}\,\mathbb{E}^{\mathbb{P}^*}[S_T - K]$$

$$g^{call}_t \geq e^{-r(T-t)}\,\mathbb{E}^{\mathbb{P}^*}[S_T] - e^{-r(T-t)}K$$

recall that $\mathbb{E}^{\mathbb{P}^*}_t\left[\frac{S_T}{B_T}\right] = \frac{S_t}{B_t} \Rightarrow S_t = \mathbb{E}^{\mathbb{P}^*}_t[S_T]\,e^{-r(T-t)}$

$$\Rightarrow g^{call}(t,S) \geq S - e^{-r(T-t)}K$$

---

recall the hedge position (i.e. to locally recall the hedge position (i.e. to locally remove risk)

$$\alpha_t = \frac{\sigma_t^g \, g_t}{\sigma_t^F \, f_t}$$

option

traded asset

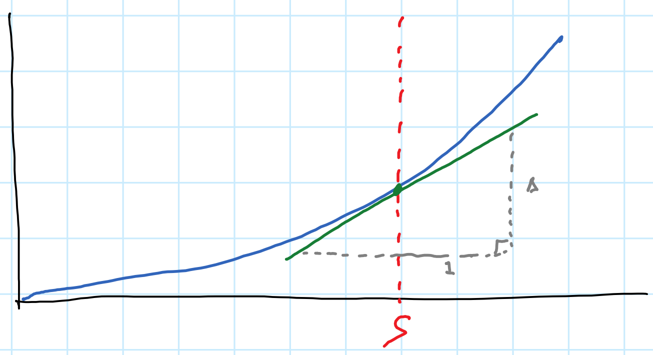
In this case:

$$\sigma_t^g \, g_t = \partial_S g(t, S_t) \cdot \sigma \cdot S_t$$

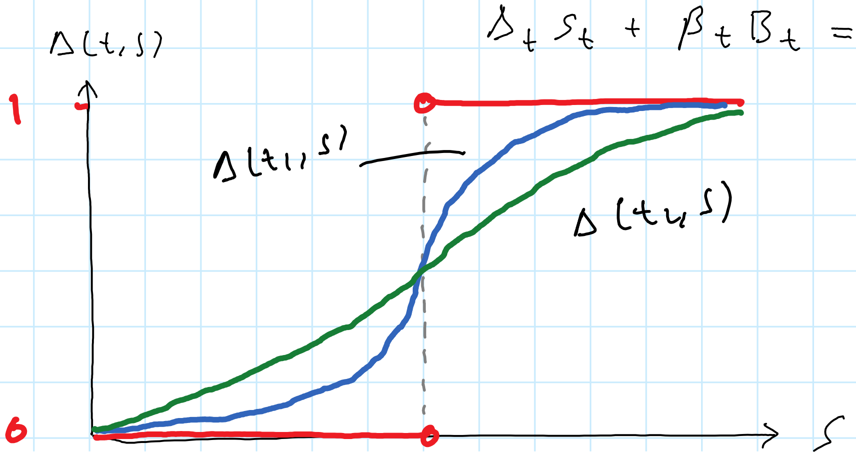
$$\sigma_t^F \, f_t = \sigma \cdot S_t$$

$$\Rightarrow \quad \boxed{\alpha_t = \partial_S g(t, S_t)} \quad - \text{ called the option's Delta}$$

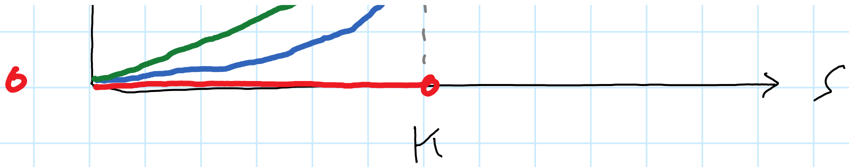
$$\Delta(t, S) = \partial_S g(t, S)$$



$$\Delta_t S_t + \beta_t B_t = g_t$$



call



$$\Delta(t,S) = \partial_S\, g(t,S)$$

$$= \partial_S\, \mathbb{E}^{\mathbb{P}^*}_{t,S}\left[(S_T - K)_+ \right] \overbrace{e^{-r(T-t)}}^{\gamma}$$

$$= \partial_S\, \mathbb{E}^{\mathbb{P}^*}\left[(S \cdot e^X - K)_+ \right] \gamma$$

$$X \underset{\mathbb{P}^*}{\sim} \mathcal{N}\left((r - \tfrac{1}{2}\sigma^2)(T-t);\ \sigma^2 (T-t) \right)$$

$$= \mathbb{E}^{\mathbb{P}^*}\left[\partial_S (S e^X - K)_+ \right] \gamma$$

$$= \mathbb{E}^{\mathbb{P}^*}\left[e^X \mathbb{1}_{S e^X > K} \right] \gamma$$

$$= \frac{1}{S} \mathbb{E}^{\mathbb{P}^*}\left[S e^X \mathbb{1}_{S e^X > K} \right] \gamma$$

$$= \mathbb{E}^{\mathbb{P}^*}\left[\frac{\gamma S_T}{S} \mathbb{1}_{S_T > K} \right]$$

$$\frac{d\hat{\mathbb{P}}}{d\mathbb{P}^*} = \frac{S_T}{S} e^{-r(T-t)} > 0 \quad a.s.$$

$$\mathbb{E}^{\mathbb{P}^*}_{t,S}\left[\frac{d\hat{\mathbb{P}}}{d\mathbb{P}^*} \right] = \frac{\mathbb{E}^{\mathbb{P}^*}_{t,S}\left[S_T \right]}{S} e^{-r(T-t)} = 1$$

$$\Rightarrow \Delta(t,S) = \mathbb{E}^{\hat{\mathbb{P}}}\left[\mathbb{1}_{S_T > K} \right] = \hat{\mathbb{P}}(S_T > K) = \Phi(d_+)$$

$$\Rightarrow \quad \Delta(t,s) = \mathbb{E}^{\hat{\mathbb{P}}}\left[\mathbb{1}_{S_T > K} \right] = \hat{\mathbb{P}}(S_T > K) = \Phi(d_+)$$

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$\underline{t=0}$ $V_0 = 0$ sold 1 of g

hold $\alpha_0 = \Delta_0$ of asset

remainder in bank account $M_0 = \beta_0 B_0$

$$\alpha_0 = \Delta_0 = \partial_S g(0, S_0)$$

$$\alpha_0 S_0 + M_0 - g_0 = 0$$

$$\Rightarrow M_0 = g_0 - \alpha_0 S_0$$

$\underline{t_1}$

$$M_0 \rightarrow M_{t_1^-} = M_0 e^{r \Delta t}$$

$$S_0 \rightarrow S_{t_1}$$

have α_0 of S but need

$$\alpha_{t_1} = \Delta_{t_1} = \partial_S g(t_1, S_{t_1})$$

$$M_{t_1^-} \rightarrow M_{t_1} = M_{t_1^-} - (\alpha_{t_1} - \alpha_0) S_{t_1}$$

$\vdots$

$$t_k \longrightarrow t_{k+1}$$

holding α_{t_k} of S update

at t_{k+1} to $\alpha_{t_{k+1}}$

$$M_{t_{k+1}} = M_{t_k} e^{r \Delta t_k} - (\alpha_{t_{k+1}} - \alpha_{t_k}) S_{t_{k+1}}$$

$$M_{t_{k+1}} = M_{t_k} e^{r\Delta t_k} - (\alpha_{t_{k+1}} - \alpha_{t_k}) S_{t_{k+1}}$$

$\vdots$

$t_{n-1} \longrightarrow t_n = T$

$\alpha_{t_{n-1}}$ of S

$M_{t_{n-1}} e^{r\Delta t}$ in bank

owe the option payoff:

e.g., $(S_{t_n} - K)_+$

$$\boxed{PnL = M_{t_{n-1}} e^{r\Delta t} + \alpha_{t_{n-1}} S_{t_n} - G(S_{t_n})}$$

This is Delta-Hedging (time-based approach)

Move-based Delta-Hedging

single call