Wednesday, October 5, 2016 2:11 PM

$k=0$ as a numeraire $\quad k=1\ldots n$

$(A_t^{(k)})_{t\geq 0}$ price processes for traded asset k

$\{(\alpha_t^{(k)})_{t\geq 0}\}_k$ trading strategy for assets

$\alpha_{t_{n-1}}^{(k)}$

$t_{n-1} \qquad t_n$

$$V_{t_n} = \sum_k \alpha_{t_{n-1}}^{(k)} A_{t_n}^{(k)} = \sum_k \alpha_{t_n}^{(k)} A_{t_n}^{(k)}$$

self-financing constraint

$$\{\alpha_{t_{n-1}}^{(k)}\}_{k=1\ldots n} \xrightarrow[t_n]{\text{at}} \{\alpha_{t_n}^{(k)}\}_{k=1\ldots n}$$

$$\boxed{\sum_k (\alpha_{t_n}^{(k)} - \alpha_{t_{n-1}}^{(k)}) A_{t_n}^{(k)} = 0}$$

$$\underbrace{(\alpha_{t_n}^{(k)} - \alpha_{t_{n-1}}^{(k)})}_{\Delta\alpha_{t_{n-1}}^{(k)}} \qquad A_{t_n}^{(k)} \to A_{t_{n-1}}^{(k)} + \Delta A_{t_{n-1}}^{(k)}$$

$$\sum_k \left(\Delta\alpha_{t_{n-1}}^{(k)} \, \Delta A_{t_{n-1}}^{(k)} + A_{t_{n-1}}^{(k)} \, \Delta\alpha_{t_{n-1}}^{(k)}\right) = 0$$

$$\xrightarrow[\Delta t \downarrow 0]{} \sum_k \left\{ d[\alpha^{(k)}, A^{(k)}]_t + A_t^{(k)} \, d\alpha_t^{(k)} \right\} = 0$$

$$V_t = \sum_k \alpha_t^{(k)} A_t^{(k)}$$

$$dV_t = \sum_k d(\alpha_t^{(k)} A_t^{(k)})$$

$$= \sum_k \left\{ d\alpha_t^{(k)} A_t^{(k)} + \alpha_t^{(k)} dA_t^{(k)} + d[\alpha^{(k)}, A^{(k)}]_t \right\}$$

if α is self-financing

$$\overset{\downarrow}{=} \sum_k \alpha_t^{(k)} dA_t^{(k)}$$

$$\boxed{dV_t = \sum_k \alpha_t^{(k)} dA_t^{(k)}}$$ for self-financing strategies

---

An arbitrage strategy $\alpha = \{(\alpha_t^{(k)})_{t\geq 0}\}_{k=1..n}$,

is a self-financing one s.t.

i) $V_0 = 0$

ii) $\exists\, t_a$ s.t. a) $\mathbb{P}(V_{t_a} \geq 0) = 1$

b) $\mathbb{P}(V_{t_a} > 0) > 0$

Model for economy:

1) $\exists$ an underlying Itô process $X = (X_t)_{t\geq 0}$,

Markov and satisfies the SDE

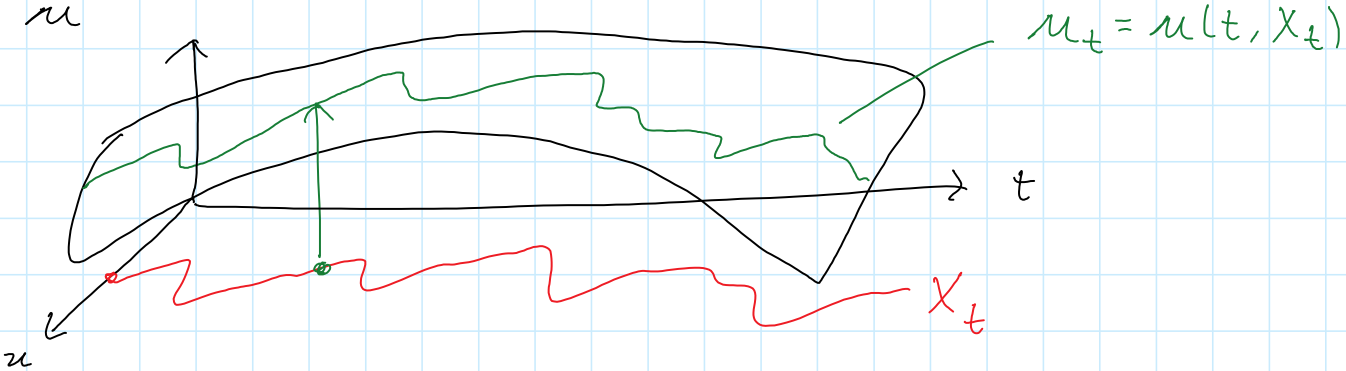
$$(1) \qquad dX_t = \mu(t, X_t)\,dt + \overbrace{\sigma(t, X_t)}^{\text{volatility}}\,dW_t$$

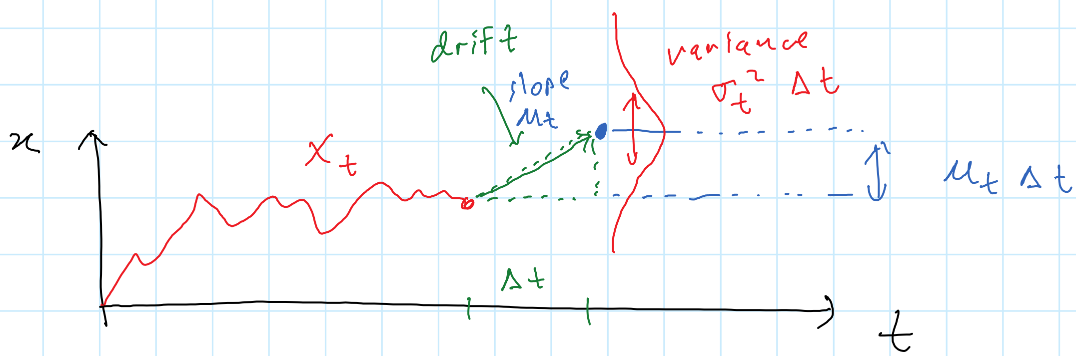
$$(1) \qquad dX_t = \underbrace{\mu(t, X_t)}_{\text{drift}} \, dt + \overbrace{\sigma(t, X_t)} \, dW_t$$

$$\mu : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$$
$$(t \, , \, x)$$

$$\sigma : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}_+ \setminus \{0\}$$

$W = (W_t)_{t \geq 0}$ is a $\mathbb{P}$-Brownian motion





Euler discretization of SDE (1)

$$X_{t_n} - X_{t_{n-1}} \stackrel{d}{=} \mu(t_{n-1}, X_{t_{n-1}}) \, \Delta t$$
$$+ \, \sigma(t_{n-1}, X_{t_{n-1}}) \sqrt{\Delta t} \; Z$$

$Z \sim N(0,1)$

$$\int_0^t W_u \, dW_u = \frac{1}{2}\left(W_t^2 - \overset{\nearrow 0}{W_0^2}\right) - \frac{1}{2} t$$

$$d(W_t^2) = 2\, dW_t \, W_t + d\underbrace{[W,W]_t}_{t}$$

$$\Rightarrow W_t \, dW_t = \frac{1}{2}\left(d(W_t^2) - dt \right)$$

$$\int_0^t W_u \, dW_u = \frac{1}{2}\left(\int_0^t d(W_t^2) - t\right)$$

$$= \frac{1}{2}\left(W_t^2 - W_0^2 - t\right)$$

$$\sum_m W_{t_{m-1}} (W_{t_m} - W_{t_{m-1}}) \qquad = \frac{1}{2}\left(W_t^2 - t\right)$$

Economic model:

1) an underlying process (Ito) $X = (X_t)_{t \geq 0}$ satisfying the SDE:

$$dX_t = \underbrace{\overbrace{\mu^X(t, X_t)}^{\mu^X_t}}_{\text{drift}} dt + \underbrace{\overbrace{\sigma^X(t, X_t)}^{\text{volatility}}}_{\sigma^X_t} dW_t$$

2) Bank account account $B = (B_t)_{t \geq 0}$ s.t.

$$dB_t = B_t\, r(t, X_t)\, dt$$

(instaneously risk-free)

$r : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}$

3) $\exists$ a traded asset (option) on X, call it $f = (f_t)_{t \geq 0}$ we assume it satisfies the SDE:

$$df_t = f_t \left(\underbrace{\overbrace{\mu^f(t, X_t)}^{\mu^f_t}}_{\substack{\text{instaneous} \\ \text{known return}}} dt + \underbrace{\overbrace{\sigma^f(t, X_t)}^{\sigma^f_t}}_{\substack{\text{instaneous} \\ \text{known volatity}}} dW_t \right)$$

4) Goal: how to value a contingent claim (option) which pays $G(X_T)$ at time T.

price process is $g = (g_t)_{0 \le t \le T}$

short claim $\xrightarrow{\text{fixed}}$ g

take $(\alpha, \beta) = (\alpha_t, \beta_t)_{t \ge 0}$ } self-financing

in F, B

1) $V_0 = 0$

2) $V_t = \alpha_t f_t + \beta_t B_t - g_t$

$$dV_t = \alpha_t \, df_t + \beta_t \, dB_t - 1 \cdot dg_t$$

self-financing condition

Assume g is Markov in X, i.e. $\exists$ a fn s.t.

$g_t = g(t, X_t)$ and $g(\,) \in C^{1,2}$

$g(\,,\,)$ is a function $\mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}$

$$\Rightarrow \quad dg_t = (\quad)\, dt + (\quad)\, dW_t$$

from Ito's lemma

write this as:

$$dg_t = g_t\left(\mu^g(t, X_t)\, dt + \sigma^g(t, X_t)\, dW_t\right)$$

$$\Rightarrow \quad dV_t = \alpha_t \, df_t + \beta_t \, dB_t - dg_t$$

$$= \alpha_t f_t (\mu^f_t \, dt + \sigma^f_t \, dW_t)$$

$$= \alpha_t f_t (\mu^f_t \, dt + \sigma^f_t \, dW_t)$$
$$+ \beta_t B_t r_t \, dt$$
$$- g_t (\mu^g_t \, dt + \sigma^g_t \, dW_t)$$

$$= (\alpha_t f_t \mu^f_t + \beta_t B_t r_t - g_t \mu^g_t) dt$$
$$+ (\alpha_t f_t \sigma^f_t - g_t \sigma^g_t) dW_t$$

locally remove risk:

$$\boxed{\alpha_t = \frac{g_t \sigma^g_t}{f_t \sigma^f_t}}$$

$$\Rightarrow \quad dV_t = (\underbrace{\alpha_t f_t \mu^f_t + \beta_t r_t B_t - g_t \mu^g_t}_{A_t}) dt$$

i.e. V has previsible drifts.

to avoid arbitrage we need $A_t = 0$

$\because \; V_0 = 0 , \quad dV_t = 0 \quad \Rightarrow \quad V_t = 0 .$

$$\Rightarrow \quad \alpha_t f_t + \beta_t B_t - g_t = 0$$

$$\Rightarrow \quad \beta_t B_t = (g_t - \alpha_t f_t)$$

Hence $A_t = 0 \Rightarrow$

$$\alpha_t f_t \mu^f_t + r_t (g_t - \alpha_t f_t) - g_t \mu^g_t = 0$$

$$\Rightarrow \quad g_t \frac{\sigma_t^g}{\sigma_t^f} \mu_t^f + r_t \left(g_t - g_t \frac{\sigma_t^g}{\sigma_t^f} \right) - g_t \mu_t^g = 0$$

$$\Rightarrow \quad \sigma_t^g \frac{\mu_t^f}{\sigma_t^f} + r_t - \frac{\sigma_t^g}{\sigma_t^f} r_t - \mu_t^g = 0$$

$$\Rightarrow \quad \boxed{\frac{\mu_t^f - r_t}{\sigma_t^f} = \frac{\mu_t^g - r_t}{\sigma_t^g}}$$

f was arbitrary hence

$$\boxed{\frac{\mu_t^g - r_t}{\sigma_t^g} = \lambda_t = \lambda(t, X_t)}$$

market-price of risk

$$dg_t = d\, g(t, X_t)$$

Ito's lemma

$$\begin{aligned} = \; & \underbrace{\partial_t g(t, X_t)\, dt + \mu^x(t, X_t)\, \partial_x g(t, X_t)\, dt + \tfrac{1}{2} (\sigma^x(t, X_t))^2\, \partial_{xx} g(t, X_t)\, dt}_{g_t \mu_t^g} \\ & + \underbrace{\sigma^x(t, X_t)\, \partial_x g(t, X_t)}_{g_t \sigma_t^g}\, dW_t \end{aligned}$$

from before: $\quad \dfrac{\mu_t^g - r_t}{\sigma_t^g} = \lambda_t$

$$\Rightarrow \quad \mu_t^g - r_t = \lambda_t \sigma_t^g$$

$$\Rightarrow \quad \mu_t^g - \lambda_t \sigma_t^g = r_t$$

$$\Rightarrow \quad \mu_t^g - \lambda_t \, \sigma_t^g = r_t$$

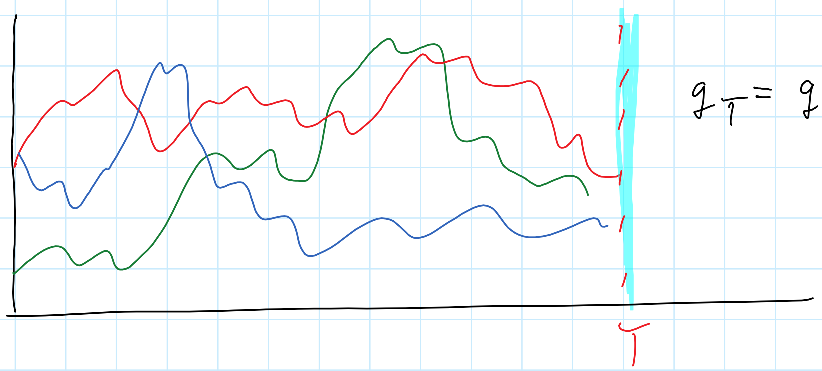
$$\Rightarrow \quad g_t \, \mu_t^g - \lambda_t \, g_t \, \sigma_t^g = r_t \, g_t$$

$$\Rightarrow \quad \partial_t \, g(t, X_t)$$

$$+ \left(\mu_t^X(t, X_t) - \lambda_t \, \sigma_t^X(t, X_t) \right) \partial_x g(t, X_t)$$

$$+ \tfrac{1}{2} \left(\sigma^X(t, X_t) \right)^2 \partial_{xx} g(t, X_t) = r(t, X_t) \, g(t, X_t)$$

this must hold $\forall$ paths of X and hence is a PDE in all of $\mathbb{R}_+ \times \mathbb{R}$



$$g_T = g(T, X_T) = G(X_T)$$

$$\boxed{\begin{aligned} (\partial_t + \mathcal{L}_t^X) \, g(t,x) &= r(t,x) \, g(t,x) \\ g(T,x) &= G(x) \end{aligned}}$$

Generalization of Black-Scholes PDE

where
$$\mathcal{L}_t^X g(t,x) = \left(\mu^X(t,x) - \lambda(t,x) \, \sigma^X(t,x) \right) \partial_x g(t,x) + \tfrac{1}{2} \left(\sigma^X(t,x) \right)^2 \partial_{xx} g(t,x)$$

infinitesimal generator of X

Feynman-Kac Theorem

suppose that $h : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}$ s.t. $h \in C^{1,2}$

$$\begin{cases} \partial_t h + \frac{1}{2}\partial_{xx} h = 0 \\ h(T,x) = H(x) \end{cases}$$

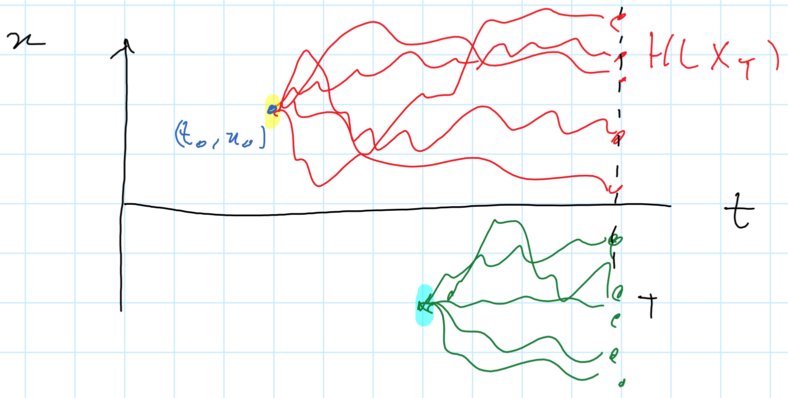
heat equation

This admits a probabilistic representation:

$$h(t,x) = \mathbb{E}^{\mathbb{P}^*}_{t,x}\left[H(X_T) \right]$$

where $\mathbb{E}^{\mathbb{P}^*}_{t,x}[\,\cdot\,]$ means $\mathbb{E}^{\mathbb{P}^*}[\,\cdot \mid X_t = x]$

and $(X_t)_{t\geq 0}$ a $\mathbb{P}^*$ - B. mtn.



define $f = (f_t)_{t\geq 0}$,

$$f_t = \mathbb{E}^{\mathbb{P}^*}\left[H(X_T) \mid \mathcal{F}^X_t \right]$$

$\times$ f is Markov in X ...

$$f_t = \mathbb{E}^{\mathbb{P}^*}\Big[H\big(\underbrace{(X_T - X_t)}_{\overset{d}{=} \sqrt{T-t}\, Z} + X_t \big) \mid \mathcal{F}^X_t \Big]$$

$$f_t = \mathbb{E}\Big[H\big(\underbrace{(X_T - X_t)}_{\stackrel{d}{=} \sqrt{T-t}\, Z} + X_t\big) \,\Big|\, \mathcal{F}_t \Big]$$

$$= \mathbb{E}^{\mathbb{P}^*}\left[H\left(\sqrt{T-t}\, Z + X_t\right) \,\middle|\, \mathcal{F}_t^X \right]$$

where $Z \underset{\mathbb{P}^*}{\sim} \mathcal{N}(0,1)$ & $\perp \mathcal{F}_t^X$

$$= \int_{-\infty}^{\infty} H\left(\sqrt{T-t}\, z + X_t\right) \frac{e^{-\frac{1}{2} z^2}}{\sqrt{2\pi}}\, dz$$

$$= f(t, X_t) \in C^{1,2}$$

Itô's lemma

$$df_t \overset{\Gamma}{=} \partial_t f(t, X_t)\, dt + 0 \cdot \partial_x f(t, X_t)\, dt$$

$$+ \tfrac{1}{2} \partial_{xx} f(t, X_t)\, dt$$

$$+ \partial_x f(t, X_t)\, dX_t$$

$$\boxed{\begin{aligned} df_t &= \left(\partial_t f(t, X_t) + \tfrac{1}{2} \partial_{xx} f(t, X_t)\right) dt \\ &\quad + \partial_x f(t, X_t)\, dX_t \end{aligned}}$$

($*$)

f is a $\mathbb{P}^*$ martingale

need to check that: i) $\mathbb{E}^{\mathbb{P}^*}[f_s \mid \mathcal{F}_t^X] = f_t$

ii) $\mathbb{E}^{\mathbb{P}^*}[|f_t|] < +\infty \;\; \forall t$

$$\mathbb{E}^{\mathbb{P}^*}[f_s \mid \mathcal{F}_t^X] = \mathbb{E}^{\mathbb{P}^*}\Big[\underbrace{\mathbb{E}^{\mathbb{P}^*}[H(X_T) \mid \mathcal{F}_s^X]}_{f_s} \,\Big|\, \mathcal{F}_t^X \Big]$$

$$= \mathbb{E}^{\mathbb{P}^*}[H(X_T) \mid \mathcal{F}_t^X]$$

$$= f_t$$

from (*) we have that

$$f_s = f_t + \int_t^s (\partial_t + \tfrac{1}{2}\partial_{xx}) f(u, X_u)\, du + \int_t^s \partial_x f(u, X_u)\, dX_u$$

$$\mathbb{E}^{\mathbb{P}^*}[\,\cdot\,|\mathcal{F}_t^X]$$

N.B.: $\mathbb{E}[\int_t^s \alpha_u\, dW_u] = 0$

$$\Rightarrow \cancel{f_t} = \cancel{f_t} + \mathbb{E}^{\mathbb{P}^*}\left[\int_t^s (\partial_t + \tfrac{1}{2}\partial_{xx}) f(u, X_u)\, du \,\Big|\, \mathcal{F}_t^X\right]$$

$s = t + \Delta t$, $\quad \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t}\left(\qquad\right)$

$$0 = \lim_{\Delta t \downarrow 0} \mathbb{E}^{\mathbb{P}^*}\left[\frac{1}{\Delta t}\int_t^{t+\Delta t} (\partial_t + \tfrac{1}{2}\partial_{xx}) f(u, X_u)\, du \,\Big|\, \mathcal{F}_t^X\right]$$

$$= \mathbb{E}^{\mathbb{P}^*}\left[(\partial_t + \tfrac{1}{2}\partial_{xx}) f(t, X_t) \,\big|\, \mathcal{F}_t^X\right]$$

F.T.C.

$$= (\partial_t + \tfrac{1}{2}\partial_{xx}) f(t, X_t)$$

holds $\forall$ paths of X $\Rightarrow$

$$\begin{cases} (\partial_t + \tfrac{1}{2}\partial_{xx}) f(t,x) = 0 \\ f(T,x) = H(x) \end{cases}$$

QED