$\underline{t=0}$ $\qquad\qquad$ $\underline{t=1}$

A_0 $\qquad\qquad$ $A_1 = A_u x + A_d (1-x)$

$x =$ Bernoulli r.v.
with success p
$\in \{0, 1\}$

$$A_0 \xrightarrow{p} A_u, \qquad A_0 \to A_d$$

$$A_0 \overset{?}{=} \mathbb{E}^{\mathbb{P}}[A_1]$$

$$= A_u p + A_d (1-p)$$

$$= A_u \mathbb{E}^{\mathbb{P}}[x] + A_d \mathbb{E}^{\mathbb{P}}[1-x]$$

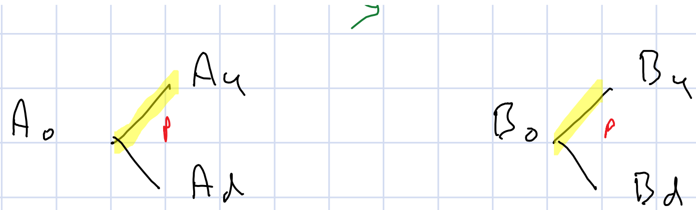
$$= A_u p + A_d (1-p)$$

utility, prospect theory, model uncertainty

(A_0, B_0) $\qquad\qquad$ $(A_1, B_1) = (A_u x + A_d (1-x), B_u x + B_d (1-x))$

$$(A_0, B_0) \xrightarrow{p} (A_u, B_u), \qquad (A_0, B_0) \to (A_d, B_d)$$

A_u $\qquad\qquad$ B_u



portfolio is (α, β) units of (A, B)

value process of this portfolio is:

$$V_0 = \alpha A_0 + \beta B_0$$

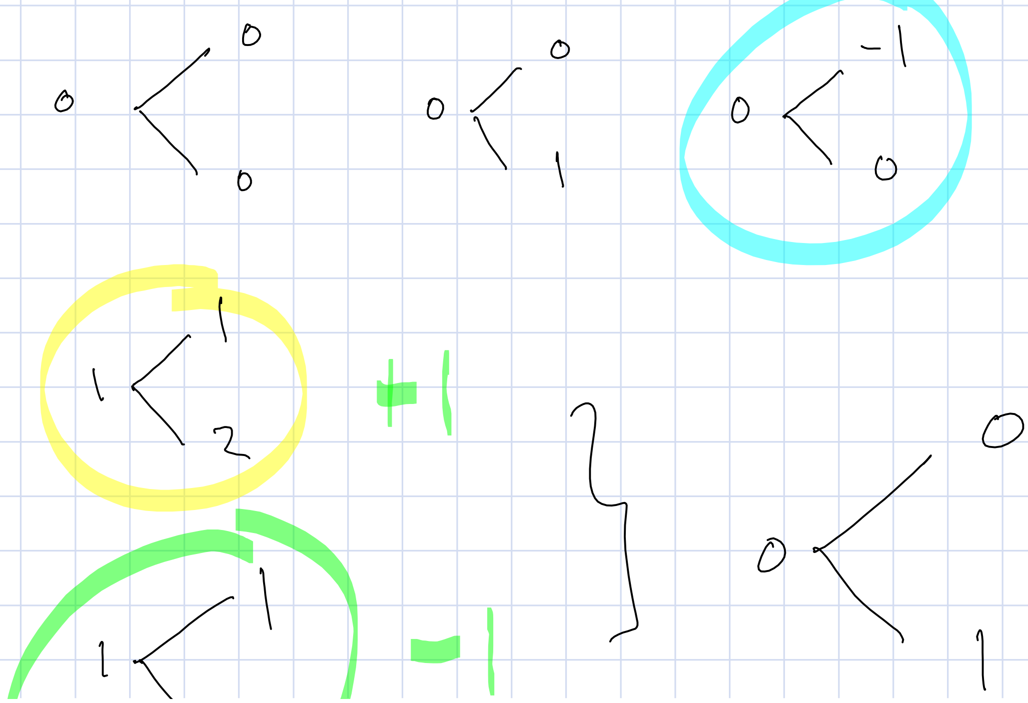
$$V_1 = \alpha A_1 + \beta B_1$$

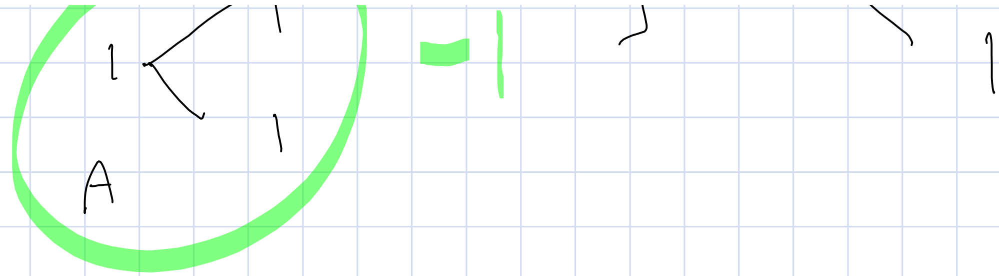
an arbitrage is a portfolio s.t.

i) $V_0 = 0$ costs nothing

ii) a) $\mathbb{P}(V_1 \geq 0) = 1$ never lose

b) $\mathbb{P}(V_1 > 0) > 0$ sometimes win.





choose (α, β) s.t. $V_0 = 0$

$$\Rightarrow \quad \alpha A_0 + \beta B_0 = 0$$

$$\Rightarrow \quad \alpha A_0 = -\beta B_0$$

assume $B_0 \neq 0$

$$\Rightarrow \quad \beta = -\alpha \frac{A_0}{B_0}$$

① ②

$$0 \left\langle \begin{array}{l} \alpha A_u + \beta B_u = \alpha \left(A_u - A_0 \dfrac{B_u}{B_0} \right) < 0 \qquad > 0 \\ \alpha A_d + \beta B_d = \alpha \left(A_d - A_0 \dfrac{B_d}{B_0} \right) > 0 \qquad < 0 \end{array} \right.$$

or

assumed $p \in (0, 1)$

② $\quad A_u > A_0 \dfrac{B_u}{B_0}$ and $A_d < A_0 \dfrac{B_d}{B_0}$

$B_0, B_u, B_d > 0$ numeraire assets

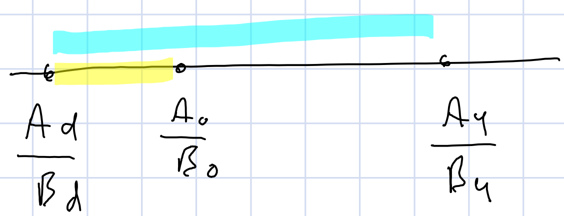
$$\Rightarrow \quad \frac{A_u}{B_u} > \frac{A_0}{B_0} \quad \text{and} \quad \frac{A_d}{B_d} < \frac{A_0}{B_0}$$

$$\boxed{\frac{A_d}{B_d} < \frac{A_0}{B_0} < \frac{A_u}{B_u}}$$

$\iff \exists$ no arbitrage

$\tilde{A}_t \triangleq \dfrac{A_t}{B_t}$ relative price process

$$\frac{A_0}{B_0} \begin{cases} A_u / B_u \\ \\ A_d / B_d \end{cases}$$



$\exists \; q \in (0,1)$ s.t.

$$\frac{A_0}{B_0} = q \frac{A_u}{B_u} + (1-q) \frac{A_d}{B_d}$$

$$\iff \tilde{A}_0 = \mathbb{E}^{\mathbb{Q}}[\tilde{A}_1]$$

$$\iff q = \frac{\frac{A_0}{B_0} - \frac{A_d}{B_d}}{\frac{A_u}{B_u} - \frac{A_d}{B_d}}$$

Fundamental Theorem of Asset Pricing:

Fundamental Theorem of Asset Pricing:

$$\boxed{\exists \text{ no arbitrage} \iff \exists\, \mathbb{Q} \sim \mathbb{P} \text{ s.t. } \tilde{A}_0 = \mathbb{E}^{\mathbb{Q}}[\tilde{A}_1]}$$

$$B_0 \begin{cases} B_0(1+r\Delta t) = B_u \\ B_0(1+r\Delta t) = B_d \end{cases}$$

Bank account

$$\frac{A_0}{B_0} = \mathbb{E}^{\mathbb{Q}}\left[\frac{A_1}{\underbrace{B_0(1+r\Delta t)}_{(B_1}}\right]$$

$$\Rightarrow \quad A_0 = \frac{1}{1+r\Delta t}\,\mathbb{E}^{\mathbb{Q}}[A_1]$$

$$\Rightarrow \quad \mathbb{E}^{\mathbb{Q}}[A_1] = (1+r\Delta t)\,A_0$$

$$\text{asset return} \triangleq \frac{\mathbb{E}^{\mathbb{Q}}[A_1] - A_0}{A_0} = r\,\Delta t$$

$\mathbb{Q}$ associated with the Bank account is called the risk-neutral measure.

$$A_0 \begin{cases} A_u \\ A_d \end{cases} \qquad B_0 \begin{cases} B_u \\ B_d \end{cases} \qquad C_0 \begin{cases} C_u \\ C_d \end{cases}$$

$$A_0 \begin{array}{l} \nearrow \\ \searrow \; A_d \end{array} \qquad B_0 \begin{array}{l} \nearrow \\ \searrow \; B_d \end{array} \qquad C_0 \begin{array}{l} \nearrow \\ \searrow \; C_d \end{array}$$

numeraire assets

$\exists$ no arb. $\iff$ $\exists\ \mathbb{Q}^B \sim \mathbb{P}$ s.t.

$$\boxed{\frac{A_0}{B_0} = \mathbb{E}^{\mathbb{Q}^B}\left[\frac{A_1}{B_1}\right]}$$

and $$\frac{C_0}{B_0} = \mathbb{E}^{\mathbb{Q}^B}\left[\frac{C_1}{B_1}\right]$$

also: $\exists\ \mathbb{Q}^C \sim \mathbb{P}$ s.t.

$$\boxed{\frac{A_0}{C_0} = \mathbb{E}^{\mathbb{Q}^C}\left[\frac{A_1}{C_1}\right]}$$

and $$\frac{B_0}{C_0} = \mathbb{E}^{\mathbb{Q}^C}\left[\frac{B_1}{C_1}\right]$$

$\mathbb{Q}^B \sim \mathbb{Q}^C$

$$\frac{A_0}{B_0} = q^B \frac{A_u}{B_u} + (1-q^B)\frac{A_d}{B_d}$$

$$\frac{C_0}{B_0}\frac{A_0}{C_0} = \underbrace{\left(q^C \frac{C_0}{B_0}\frac{B_u}{C_u}\right)}_{q^*}\frac{A_u}{B_u} + \underbrace{\left((1-q^C)\frac{C_0}{B_0}\frac{B_d}{C_d}\right)}_{q^{**}}\frac{A_d}{B_d}$$

$$q^{\alpha} + q^{\alpha *} = \frac{C_0}{B_0}\left(q^{c} \frac{B_u}{C_u} + (1-q^{c}) \frac{B_d}{C_d} \right) = 1$$

$$\mathbb{E}^{\mathbb{Q}^c}\left[\frac{B_1}{C_1} \right] = \frac{B_0}{C_0}$$

$$\Rightarrow \quad q^{B} = q^{c} \frac{B_u / B_0}{C_u / C_0} \quad \text{and}$$

$$(1-q^{B}) = (1-q^{c}) \frac{B_d / B_0}{C_d / C_0}$$

$$\mathbb{Q}^{B}(\cdot) = \mathbb{Q}^{c}(\cdot) \frac{B_1(\cdot) / B_0}{C_1(\cdot) / C_0}$$

$$\frac{A_0}{B_0} = \mathbb{E}^{\mathbb{Q}^B}\left[\frac{A_1}{B_1} \right]$$

$$= \mathbb{E}^{\mathbb{Q}^B}\left[\frac{A_1}{C_1} \frac{C_1}{B_1} \right]$$

$$= \mathbb{E}^{\mathbb{Q}^c}\left[\frac{A_1}{C_1} \right] \cdot \frac{C_0}{B_0}$$

$$\Rightarrow \quad \frac{A_0}{C_0} = \mathbb{E}^{\mathbb{Q}^c}\left[\frac{A_1}{C_1} \right]$$

we assume $\exists$ a numeraire B.

FTAP: $\exists$ no arb $\iff$ $\exists$ $\mathbb{Q}^B \sim \mathbb{P}$ s.t. for all traded assets A we have:

$$\frac{A_t}{B_t} = \mathbb{E}^{\mathbb{Q}^B}\left[\frac{A_u}{B_u} \,\middle|\, \mathcal{F}_t \right]$$

$\forall \; u \geq t$.

$\mathbb{F} = (\mathcal{F}_u)_{u \geq 0}$ is the filtration on which assets are measurable.

i.e. the relative price process is a martingale. $\tilde{A}_t = \mathbb{E}^{\mathbb{Q}^B}[\tilde{A}_u \mid \mathcal{F}_t]$

Cox, Ross, Rubenstein model (CRR)

$$B_{t_n} = B_{t_{n-1}} e^{r \Delta t}, \quad t_n = n \Delta t, \quad n = 0, 1, 2, \ldots$$

$$A_{t_n} = A_{t_{n-1}} e^{\sigma \sqrt{\Delta t}\, x_n}, \quad x_1, x_2, \ldots \text{ iid.}$$

Bernoulli ± 1

$$\mathbb{P}(x_1 = +1) = \frac{1}{2}\left(1 + \frac{\mu - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right) \in (0,1)$$

$$\frac{\cancel{A_{t_n}}}{\cancel{B_{t_n}}} = q \, \frac{\cancel{A_{t_n}} e^{\sigma\sqrt{\Delta t}}}{\cancel{B_{t_n}} e^{r \Delta t}} + (1-q) \frac{\cancel{A_{t_n}} e^{-\sigma\sqrt{\Delta t}}}{\cancel{B_{t_n}} e^{r\Delta t}}$$

$$\Rightarrow \quad q = \frac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} \in (0,1)$$

$$\sigma > 0, \quad 0 < r \Delta t < \sigma\sqrt{\Delta t} \Rightarrow 0 < r < \frac{\sigma}{\sqrt{\Delta t}}$$

to avoid arbitrage.

$$e^{x} \sim 1 + x + \tfrac{1}{2}x^2 + \dots \qquad \sigma\sqrt{\Delta t} + (r - \tfrac{1}{2}\sigma^2)\Delta t + \dots$$

$$q = \frac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} = \frac{\overbrace{(\cancel{1} + r\Delta t) - (\cancel{1} - \sigma\sqrt{\Delta t} + \tfrac{1}{2}\sigma^2\Delta t) + \dots}^{\sigma\sqrt{\Delta t} + (r - \frac{1}{2}\sigma^2)\Delta t + \dots}}{\underbrace{(\cancel{1} + \sigma\sqrt{\Delta t} + \cancel{\tfrac{1}{2}\sigma^2\Delta t}) - (\cancel{1} - \sigma\sqrt{\Delta t} + \cancel{\tfrac{1}{2}\sigma^2\Delta t}) + \dots}_{2\sigma\sqrt{\Delta t} + \dots}}$$

$$= \frac{1}{2} + \frac{1}{2}\,\frac{r - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t} + \dots$$

$$\log\left(\frac{A_{t_n}}{A_0}\right) = y_{t_n} = \sigma\sqrt{\Delta t}\sum_{m=1}^{n} x_m$$

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left\{ i u \sigma\sqrt{\Delta t}\sum_{m=1}^{n} x_m \right\}\right]$$

$$= \mathbb{E}^{\mathbb{P}}\left[\prod_{m=1}^{n} e^{iu\sigma\sqrt{\Delta t}\,x_m}\right]$$

$$= \prod_{m=1}^{n} \mathbb{E}^{\mathbb{P}}\left[e^{iu\sigma\sqrt{\Delta t}\,x_m}\right]$$

$$= \left(\mathbb{E}^{\mathbb{P}}\left[e^{iu\sigma\sqrt{\Delta t}\,x_1}\right]\right)^n$$

$$\hookrightarrow \mathbb{E}^{\mathbb{P}}\left[1 + iu\sigma\sqrt{\Delta t}\,x_1 - u^2\tfrac{1}{2}\sigma^2\Delta t\,x_1^2 + \dots\right]$$

$$\mathbb{E}^{\mathbb{P}}[x_1] = 2p - 1 = \frac{\mu - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}$$

$$\mathbb{E}^{\mathbb{P}}[x_1^2] = 1$$

$$= 1 + \left(iu(\mu - \tfrac{1}{2}\sigma^2) - \tfrac{1}{2}\sigma^2 u^2\right)\Delta t + \dots$$

$$\left(1 + (\;)\frac{T}{n}\right)^n \xrightarrow[n \to \infty]{} e^{(\;)T}$$

$n \to \infty$

$$\boxed{\mathbb{E}^{\mathbb{P}}[e^{iu\,y_T}] \xrightarrow[n\to\infty]{} e^{iu(\mu-\frac{1}{2}\sigma^2)T - \frac{1}{2}\sigma^2 T u^2}}$$

$$\Rightarrow \quad y_T \underset{\mathbb{P}}{\sim} \mathcal{N}\left((\mu - \tfrac{1}{2}\sigma^2)T \,;\, \sigma^2 T \right)$$

$$\underset{\mathbb{Q}}{\sim} \mathcal{N}\left((r - \tfrac{1}{2}\sigma^2)\,T \,;\, \sigma^2 T \right)$$