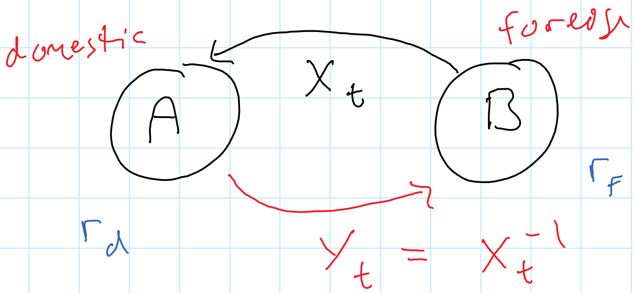
Wednesday, November 25, 2015 2:08 PM

Foreign Exchange Markets

FX



1 unit of B

$\rightarrow$ X units of A

$$\frac{dB_t^d}{B_t^d} = r_d\, dt$$

domestic bank account $B_t^d = e^{r_d t}$

Foreign " " $B_t^f = e^{r_f t}$ $\Rightarrow \dfrac{dB_t^f}{B_t^f} = r_f\, dt$

assume $\dfrac{dX_t}{X_t} = \mu\, dt + \sigma\, dW_t^{\mathbb{P}}$

domestic bank as numeraire asset - $\mathbb{Q}_d$

$\hat{B}^f$ is the domesticised foreign bank account

$$\hat{B}_t^f = X_t B_t^f$$

is identical to a domestic asset.

hence: $\tilde{B}^f = \dfrac{\hat{B}^f}{B^d}$ must be a $\mathbb{Q}_d$ - martingale (to avoid arbitrage)

so $\exists\ \lambda$ s.t. $W^{\mathbb{Q}_d}$ satisfying

$$dW_t^{\mathbb{Q}_d} = -\lambda_t\, dt + dW_t^{\mathbb{P}}$$

is a $\mathbb{Q}_d$ - Brownian.

$$\frac{d\mathbb{Q}_d}{d\mathbb{P}} = \mathcal{E}\left(\int_0^t \lambda_s \, dW_s^{\mathbb{P}} \right)$$

$$\Rightarrow \quad \frac{dX_t}{X_t} = (\mu + \sigma \lambda_t)\, dt + \sigma \, dW_t^{\mathbb{Q}_d}$$

$$d\tilde{B}_t^f = d\left(\frac{X_t \, B_t^f}{B_t^d} \right)$$

$$= dX_t \cdot \left(\frac{B_t^f}{B_t^d} \right) + X_t \, d\left(\frac{B_t^f}{B_t^d} \right)$$

$$+ \, d\left[X, \frac{B^f}{B^d} \right]_t$$

$\hookrightarrow$ 0 b/c B^f & B^d are differentiable.

$$= \left((\mu + \sigma \lambda_t)\, dt + \sigma \, dW_t^{\mathbb{Q}_d} \right) X_t \cdot \left(\frac{B_t^f}{B_t^d} \right)$$

$$+ \, X_t \, (r_f - r_d) \frac{B_t^f}{B_t^d} \, dt$$

$$\frac{d\tilde{B}_t^f}{\tilde{B}_t^f} = \underbrace{(\mu + \sigma \lambda_t + r_f - r_d)}_{=0} \, dt + \sigma \, dW_t^{\mathbb{Q}_d}$$

$= 0$ since $\tilde{B}^f$ is a $\mathbb{Q}^d$ - martingale

$$\Rightarrow \quad \lambda_t = \frac{r_d - r_f - \mu}{\sigma}$$

$$\boxed{\frac{dX_t}{X_t} = (r_d - r_f)\,dt + \sigma\, dW_t^{Q_d}}$$

As if the FX is a traded asset and pays dividends of r_f.

$$\frac{dY_t}{Y_t} = (r_f - r_d)\,dt - \sigma\, dW_t^{Q_f}$$

$$Y_t = X_t^{-1} \Rightarrow X_t = Y_t^{-1} = g(Y_t), \quad g(y) = y^{-1}$$

Ito's quotient rule ...

$$\frac{dX_t}{X_t} = \underbrace{\frac{d(1)}{1}}_{0} - \frac{dY_t}{Y_t} + \frac{d[Y,Y]_t}{Y_t^2} - \underbrace{\frac{d[1,Y]_t}{1\cdot Y_t}}_{0}$$

$$= (r_d - r_f)\,dt + \sigma\, dW_t^{Q_F}$$

$$+ \sigma^2\, dt$$

$$\Rightarrow \boxed{\frac{dX_t}{X_t} = (r_d - r_f + \sigma^2)\,dt + \sigma\, dW_t^{Q_f}}$$

$$\frac{dQ_F}{dQ_d} = \frac{\hat{B}_{f,T} / \hat{B}_{f,0}}{B_{d,T} / B_{d,0}}$$

$$\eta_t = \mathbb{E}_t^{Q_d}\left[\frac{dQ_F}{dQ_d}\right] \quad \text{is a } Q_d\text{-mtg}$$

$$= \frac{\hat{B}_{f,t} / \hat{B}_{f,0}}{B_{d,t} / B_{d,0}}$$

from before

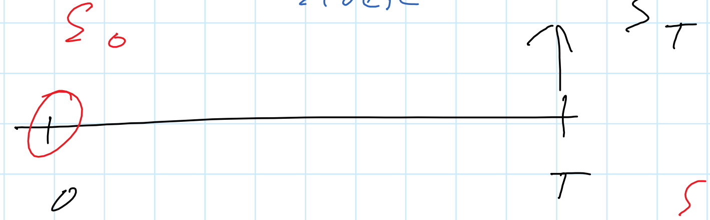
$$\frac{d\eta_t}{\eta_t} = \frac{d\left(\frac{X_t \, B_{f,t}}{B_{d,t}}\right)}{\left(\frac{X_t B_{f,t}}{B_{d,t}}\right)} = \sigma \, dW_t^{Q_d}$$

$$dW_t^{Q_f} = -\sigma \, dt + dW_t^{Q_d}$$

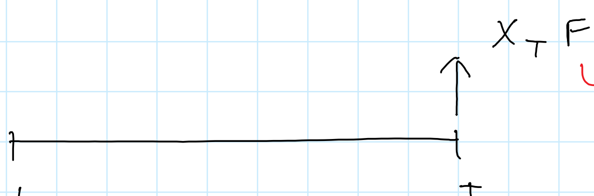
since, $\frac{dX_t}{X_t} = (r_d - r_f) \, dt + \sigma \, dW_t^{Q_d}$

$$= (r_d - r_f + \sigma^2) \, dt + \sigma \, dW_t^{Q_f}$$

For a traded stock



$$\frac{g_0}{S_0} = \mathbb{E}^{Q_S}\left[\frac{g_T}{S_T}\right] = 1$$



$$\frac{g_t}{B_{d,t}} = \mathbb{E}_t^{Q_d}\left[\frac{X_T . F}{B_{d,T}}\right]$$

$$\frac{g_t}{B_{d,t}} = \mathbb{E}_t \left[\frac{X_T \cdot F}{B_{d,T}} \right]$$

$$\Rightarrow \quad g_t = e^{-r_d (T-t)} \cdot \mathbb{E}_t^{\mathbb{Q}^d} [X_T] \cdot F$$

and $\quad X_T = X_t \, e^{[(r_d - r_f) - \frac{1}{2}\sigma^2](T-t) + \sigma (W_T^{\mathbb{Q}^d} - W_t^{\mathbb{Q}^d})}$

$$\overset{d}{=} X_t \, e^{[(r_d - r_F) - \frac{1}{2}\sigma^2](T-t) + \sigma \sqrt{T-t}\, Z^{\mathbb{Q}^d}}$$

$$Z^{\mathbb{Q}^d} \underset{\mathbb{Q}^d}{\sim} \mathcal{N}(0,1)$$

$$\Rightarrow \quad \mathbb{E}_t^{\mathbb{Q}^d} [X_T] = X_t \, e^{[(r_d - r_f) - \frac{1}{2}\sigma^2](T-t)} \cdot \mathbb{E}^{\mathbb{Q}^d} \left[e^{\sigma \sqrt{T-t}\, Z^{\mathbb{Q}^d}} \right]$$

$$= X_t \, e^{(r_d - r_F)(T-t)}$$

$$\Rightarrow \quad g_t = F \, e^{-r_F (T-t)} \, X_t$$

Forward FX-rate is an exchange rate agree on today for exchange in the future.

value is $\quad (X_T - K)\, F \quad$ at $\quad T$.
(g)

$$g_t = X_t \, F \cdot e^{-r_F (T-t)} - K F \, e^{-r_d (T-t)}$$

forward FX rate is the K in above which makes the contract worthless

$$F_t(T) = X_t \, e^{(r_d - r_F)(T-t)}$$

$$= \mathbb{E}_t^{\mathbb{Q}_d}[\, X_T \,]$$

why?

$$\mathbb{Q}_d \to \mathbb{Q}_d^T$$

$$P_t^d(T) \leftarrow \frac{g_t}{B_{d,t}} = \mathbb{E}_t^{\mathbb{Q}_d}\left[\frac{X_T - K}{B_{d,T}} \right] \times \text{Notional}$$

$$B_{d,T} \to P_T(T) = 1$$

choose K s.t.

$$0 = \mathbb{E}_t^{\mathbb{Q}_d}\left[\frac{X_T}{B_{d,T}} \right] - F_t(T)\, \mathbb{E}_t^{\mathbb{Q}_d}\left[\frac{1}{B_{d,T}} \right]$$

$$1 \to P_T^d(T)$$

$$\Rightarrow \quad F_t(T) = \Bigg(\underbrace{\mathbb{E}_t^{\mathbb{Q}_d}\left[\frac{1}{B_{d,T}} \right]}_{\frac{P_t^d(T)}{B_t}} \Bigg)^{-1} \cdot \mathbb{E}_t^{\mathbb{Q}_d}\left[\frac{X_T}{B_{d,T}} \right]$$

$$\Rightarrow \quad 0 = \mathbb{E}_t^{\mathbb{Q}_d^T}[\, X_T \,] - F_t(T)$$

$$\boxed{F_t(T) = \mathbb{E}_t^{\mathbb{Q}_d^T}[\, X_T \,]}$$

Forward exchange rate

call option on FX ?

notional

pays $(X_T - K)_+ N$ @ T.

$\hookrightarrow 1$

$$\frac{g_t}{B_{d,t}} = \mathbb{E}_t^{\mathbb{Q}_d}\left[\frac{(X_T - K)_+}{B_{d,T}}\right]$$

$$\Rightarrow \quad g_t = e^{-r_d(T-t)} \overset{+\alpha(T-t)}{} \left(\mathbb{E}^{\mathbb{Q}_d}\left[(X_T - K)_+\right]\right) \times e^{-\alpha(T-t)}$$

$$X_T \overset{d}{=} X_t \, e^{(\overbrace{(r_d - r_f)}^{\alpha} - \frac{1}{2}\sigma^2)(T-t) + \sigma\sqrt{T-t}\, Z^{\mathbb{Q}_d}}$$

$$Z^{\mathbb{Q}_d} \underset{\mathbb{Q}_d}{\sim} \mathcal{N}(0,1)$$

$$\Rightarrow \quad g_t = e^{-r_f(T-t)}\left(X_t\, \Phi(d_+) - K e^{-(r_d - r_f)(T-t)}\, \Phi(d_-)\right)$$

$$d_\pm = \frac{\log(X_t/K) + ((r_d - r_f) \pm \frac{1}{2}\sigma^2)(T-t)}{\sqrt{\sigma^2 (T-t)}}$$

$$(X_T - K)_+ N = \underbrace{X_T N \mathbb{1}_{X_T > K}}_{L_T} - \underbrace{K N \mathbb{1}_{X_T > K}}_{M_T}$$

$$\frac{M_t}{P_t^d(T)} = \mathbb{E}_t^{\mathbb{Q}_d^T}\left[\frac{\mathbb{1}_{X_T > K}}{P_T^d(T)}\right] \cdot K N$$

$\hookrightarrow 1$

$$\frac{X_T P_T^f(T)}{P_T^d(T)}$$

$$\Rightarrow \quad M_t = K N \, P_t^d(T)\, \mathbb{Q}_d^T\left((X_T) > \frac{K \cdot 1}{1} \right)$$

$$\beta_t = \frac{X_t\, P_t^f(T)}{P_t^d(T)} \text{ is a } \mathbb{Q}_d^T\text{-martingale}$$

$$\frac{d\beta_t}{\beta_t} = \sigma\, dW_t^{\mathbb{Q}_d^T}, \qquad \text{(under deterministic interest rate)}$$

$$\Rightarrow \quad \beta_T = \beta_t \cdot e^{-\frac{1}{2}\sigma^2 \underbrace{\tau}_{T-t} + \sigma\left(W_T^{\mathbb{Q}_d^T} - W_t^{\mathbb{Q}_d^T}\right)}$$

$$\overset{d}{=} \beta_t \exp\left\{-\tfrac{1}{2}\sigma^2\tau + \sigma\sqrt{\tau}\, Z^{\mathbb{Q}_d^T}\right\}$$

$$Z^{\mathbb{Q}_d^T} \underset{\mathbb{Q}_d^T}{\sim} \mathcal{N}(0,1)$$

$$\Rightarrow \quad M_t = K\, N \cdot P_t^d(T)\, \Phi(d_-)$$

$$d_- = \frac{\log(\beta_t/K) - \frac{1}{2}\sigma^2\tau}{\sqrt{\sigma^2\tau}}$$

$$\frac{L_T}{\underbrace{\hat{P}_t^f(T)}_{P_t^f(T)\, X_t}} = \mathbb{E}_t^{\mathbb{Q}_f^T}\left[\frac{\cancel{X_T}\, \mathbb{1}_{X_T > K}}{\cancel{X_T} \cdot \underbrace{P_T^f(T)}_{1}}\right] \cdot N$$

$$\frac{X_T\, P_T^f(T)}{P_T^d(T)}$$

$$\Rightarrow \quad L_T = P_t^f(T)\, X_t \cdot N \cdot \mathbb{Q}_f^T(X_T > K)$$

$$[X] \sim \frac{\text{domes}}{\text{foreign}}$$

$$\frac{d\beta_t}{\beta_t} = \sigma\, dW_t^{\mathbb{Q}_d^T}$$

$$\frac{d\mathbb{Q}_F^T}{d\mathbb{Q}_d^T} = \frac{\hat{P}_T^f(T) / \hat{P}_0^f(T)}{P_T^d(T) / P_0^d(T)}$$

$$P_t^f(T)\, X_t$$

$$\eta_t = \mathbb{E}_t^{\mathbb{Q}_d^T}\left[\frac{d\mathbb{Q}_F^T}{d\mathbb{Q}_d^T}\right] = \frac{\hat{P}_t^f(T)}{P_t^d(T)} \cdot c$$

deterministic IR....

$$\frac{d\eta_t}{\eta_t} = \sigma\, dW_t^{\mathbb{Q}_d^T}$$

$$dW_t^{\mathbb{Q}_F^T} = -\sigma\, dt + dW_t^{\mathbb{Q}_d^T}$$

$\Rightarrow$

$$\frac{d\beta_t}{\beta_t} = \sigma^2\, dt + \sigma\, dW_t^{\mathbb{Q}_F^T}$$

$$\Rightarrow \quad \beta_T \overset{d}{=} \beta_t\, e^{+\frac{1}{2}\sigma^2 \tau + \sigma\sqrt{\tau}\, Z^{\mathbb{Q}_F^T}}$$

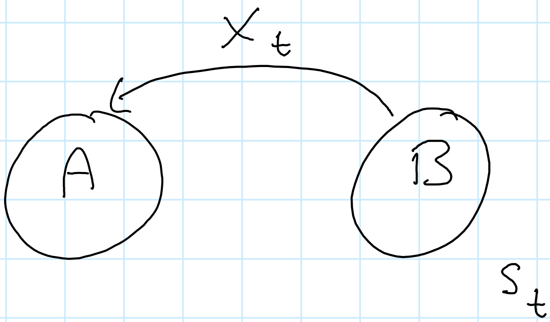
$$Z^{\mathbb{Q}_F^T} \underset{\mathbb{Q}_F^T}{\sim} \mathcal{N}(0,1)$$

$$L_t = P_t^f(T)\, X_t \cdot N\, \Phi(d_+)$$

$$d_+ = \frac{\log(\beta_t/K) + \frac{1}{2}\sigma^2 \tau}{\sqrt{\sigma^2 \tau}}$$

$$g_t = \left(P_t^f(\tau)\, X_t\, \Phi(d_+) - P_t^d(\tau)\, K\, \Phi(d_-) \right) N$$

$$d_\pm = \frac{\log \frac{X_t P_t^f(\tau)}{K P_t^d(\tau)} \pm \frac{1}{2}\sigma^2 \tau}{\sqrt{\sigma^2 \tau}}$$



quantos : options on foreign stock struck in foreign / domestic dollars

$$(S_T - K)_+ \, x$$

$$(X_T S_T - K)_+$$

$$(S_T - Y_T K)_+ \, X_T$$

$$\begin{aligned}
\frac{dS_t}{S_t} &= \mu \, dt + \eta \, dW_t^{(2),\mathbb{P}} \\
&= r_f \, dt + \eta \, dW_t^{(2),\mathbb{Q}_f} \\
&= r_f \, dt + \eta \left(-\rho \sigma \, dt + dW_t^{(2),\mathbb{Q}_d} \right) \\
&= (r_f - \rho \sigma \eta) \, dt + \eta \, dW_t^{(2),\mathbb{Q}_d}
\end{aligned}$$