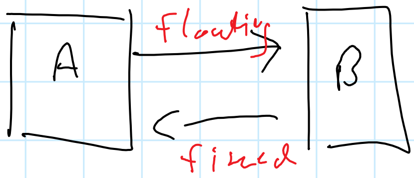
Swap options or Swaptions



Payer swaption: option to enter into the payer IRS, $(T_0, T_1, \ldots T_n)$, for a fixed rate of F, (strike) at time $T < T_0$

@ T exercise if $V_T^{fl} > V_T^{fix}$

and so $g_T = V_T^{fl} - V_T^{fix}$

don't exercise if $V_T^{fl} \leq V_T^{fix}$

$g_T = 0$

in all $g_T = (V_T^{fl} - V_T^{fix})_+$

(set $N=1$) $= \Big[P_T(T_0) - P_T(T_n) - \sum_{k=1}^{n} P_T(T_k) \cdot F \cdot \Delta T_k \Big]_+$

$(T = T_0)$ $= \Big(1 - \sum^{n} c_k P_T(T_k) \Big)$

$(T = T_0)$

$$= \left(1 - \sum_{k=1}^{n} c_k \, P_T(T_k)\right)_+$$

$$c_k = \begin{cases} F \, \Delta T_k, & k = 1, \dots, n-1 \\ (1 + F \, \Delta T_k), & k = n \end{cases}$$

recall that
(in Vasicek model)

$$P_T(T_k) = e^{\overbrace{A(T;T_k)}^{a_k} - \overbrace{B(T;T_k)}^{b_k} r_T}$$

$$h(r) = 1 - \sum_{k=1}^{n} c_k \, e^{a_k - b_k r}$$

clearly monotically increasing in r.

$\therefore \; \exists$ a unique $r^* \in \mathbb{R}$ s.t. $h(r^*) = 0$

$$\Leftrightarrow \sum_k c_k \, e^{a_k - b_k r^*} = 1$$

$$g_T = (h(r_T))_+$$

$$= (h(r_T)) \, \mathbb{1}_{r_T > r^*}$$

$$= \Big(1$$

$$- \sum_k c_k \, e^{a_k - b_k r_T} \Big) \mathbb{1}_{r_T > r^*}$$

$$= \left(\sum_k c_k \, e^{a_k - b_k r^*} - \sum_k c_k \, e^{a_k - b_k r_T} \right) \mathbb{1}_{r_T > r^*}$$

$$= \Big(\; e^{a_k - b_k r^*} \qquad e^{a_k - b_k r_T} \Big) \; \mathbb{1}$$

$$= \sum_k c_k \Big(\underbrace{e^{a_k - b_k r^*}}_{P_T(T_k;\, r^*)} - \underbrace{e^{a_k - b_k r_T}}_{P_T(T_k;\, r_T)} \Big) \mathbb{1}_{r_T > r^*}$$

$$\mathbb{1}_{r_T > r^*} = \mathbb{1}_{-b_k r_T < -b_k r^*}$$

$$= \mathbb{1}_{a_k - b_k r_T < a_k - b_k r^*}$$

$$= \mathbb{1}_{e^{a_k - b_k r_T} < e^{a_k - b_k r^*}}$$

$$= \mathbb{1}_{P_T(T_k;\, r_T) < P_T(T_k;\, r^*)}$$

$$g_T = \sum_k c_k \Big(\underbrace{P_T(T_k;\, r^*)}_{\ell} - \underbrace{P_T(T_k;\, r_T)}_{A_T} \Big) \mathbb{1}_{\underbrace{P_T(T_k;\, r_T)}_{A_T} < \underbrace{P_T(T_k;\, r^*)}_{\ell}}$$

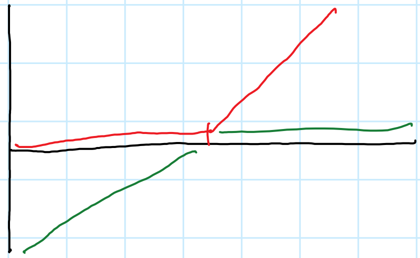
$$= \sum_k c_k \Big(P_T(T_k;\, r^*) - P_T(T_k;\, r_T) \Big)_+$$

decomposes swaptions into a strip of put options on the individual bonds which make up the cash-flow.

$$g_t = \sum_k c_k \, V_t^{put}(T, T_k;\, r^*)$$

put-call parity

NB: $(S-K)_+ - (K-S)_+ = S-K$



$$\mathbb{E}_t^{\mathbb{Q}}\left[\frac{(S_T-K)_+}{B_T} - \frac{(K-S_T)_+}{B_T}\right] = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{S_T-K}{B_T}\right]$$

$$\Rightarrow \underbrace{\mathbb{E}_t^{\mathbb{Q}}\left[\frac{(S_T-K)_+}{B_T}\right]}_{\frac{V_t^{call}}{B_t}} - \underbrace{\mathbb{E}_t^{\mathbb{Q}}\left[\frac{(K-S_T)_+}{B_T}\right]}_{\frac{V_t^{put}}{B_t}} = \frac{S_t}{B_t} - K\,\frac{P_t(T)}{B_t}$$

$$\mathbb{E}_t^{\mathbb{Q}}\left[\frac{K}{B_T}\right] = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{P_T(T)}{B_T}\right]K$$

$$= K\,\frac{P_t(T)}{B_t}$$

$$\Rightarrow \boxed{V_t^{call} - V_t^{put} = S_t - K\,P_t(T)}$$

put-call parity

recall $g_T = \left(V_T^{fl} - V_T^{fix} \right)_+$ $\quad$ $\underbrace{F \sum_k P_T(T_k)\, \Delta T_k}$

$$= \left(\boxed{\frac{P_T(T_0) - P_T(T_n)}{\sum_k P_T(T_k)\, \Delta T_k}} - F \right)_+ \left(\sum_k P_T(T_k)\, \Delta T_k \right)$$

S_T – swap-rate on day T.

$$= \left(S_T - F \right)_+ A_T$$

$A_t = \sum_k P_t(T_k)\, \Delta T_k$ – annuity

$$\frac{g_t}{A_t} = \mathbb{E}_t^{\mathbb{Q}^A} \left[\frac{(S_T - F)_+ A_T}{A_T} \right]$$

$$\Rightarrow \quad g_t = A_t\, \mathbb{E}^{\mathbb{Q}^A} \left[(S_T - F)_+ \right]$$

$$S_t = \frac{P_t(T_0) - P_t(T_n)}{\sum_k P_t(T_k)\, \Delta T_k} = \frac{P_t(T_0) - P_t(T_n)}{A_t}$$

assume (reasonable) $P_t(T_0) > P_t(T_n)$

$\Rightarrow$ S_t is a $\mathbb{Q}^A$ - martingale!

and so S_t must satisfy SDE of the type:

$$d S_t = \quad \sigma \quad dW^A$$

$$\frac{dS'_t}{S'_t} = \sigma_t \, dW_t^A$$

$\mathbb{Q}^A$ - B.motion.

For some process $\mathcal{F}$ - adapted process σ.

Log-normal swap-rate model

(LSM) assumes σ_t is deterministic

then,

$$S'_T = S'_t \exp\left\{ -\frac{1}{2} \int_t^T \sigma_s^2 \, ds + \int_t^T \sigma_s \, dW_s^A \right\}$$

$$\stackrel{d}{=} S'_t \exp\left\{ -\frac{1}{2} \Sigma^2 + \Sigma \cdot Z^A \right\}$$

$$Z^A \underset{\mathbb{Q}^A}{\sim} \mathcal{N}(0,1)$$

$$\Sigma^2 = \int_t^T \sigma_s^2 \, ds$$

$$\Rightarrow \quad g_t = A_t \, \mathbb{E}_t^{\mathbb{Q}^A}\left[\left(S'_T - F \right)_+ \right]$$

$$\boxed{g_t = A_t \left(S'_t \, \Phi(d_+) - F \, \Phi(d_-) \right)}$$

$$d_\pm = \frac{\log(S'_t / F) \pm \frac{1}{2} \Sigma^2}{}$$

Σ

most actively traded swaptions are when $F = S'_t$ (at-the-money).

B-S reminder

$$\frac{dS_t}{S_t} = r\,dt + \sigma\,dW_t \Rightarrow S_T = S_t\, e^{(r-\frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)}$$

$$\overset{d}{=} S_t\, e^{\overset{\nearrow 0}{r}(T-t) - \frac{1}{2}\Sigma^2 + \Sigma Z}$$

$$Z \underset{\mathbb{Q}}{\sim} \mathcal{N}(0,1), \quad \Sigma = \sigma^2 (T-t)$$

$$\hookrightarrow \int_t^T \sigma_s^2\, ds$$

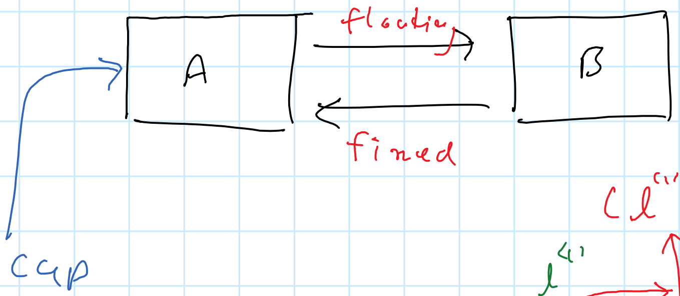
$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\overset{\nearrow 0}{r}(T-t)} (S_T - K)_+\right]$$

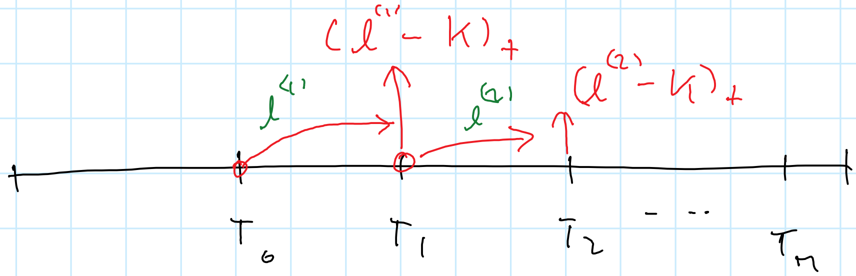
$$= S_t\, \Phi(d_+) - \overset{\nearrow F}{K}\, e^{-\overset{\nearrow 0}{r}(T-t)}\, \Phi(d_-)$$

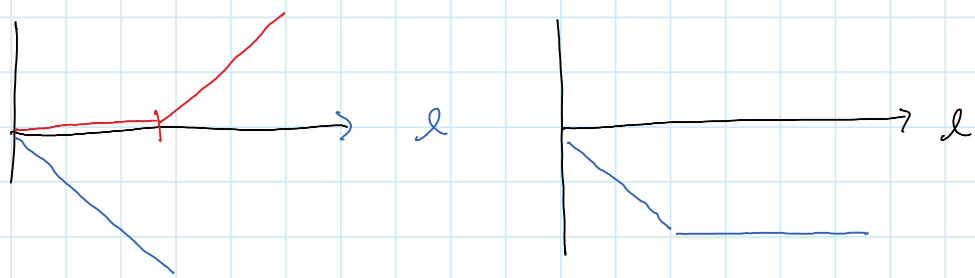
$S_t \swarrow S'_t$

$$d_\pm = \frac{\log(S_t/K) + \overset{\nearrow 0}{r}(T-t) \pm \frac{1}{2}\Sigma^2}{\Sigma}$$

Interest Rate Caps & Floors



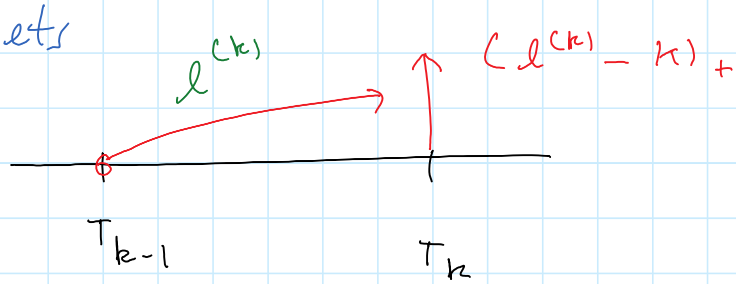




$$g_t = \sum_k g_t^{(k)}$$

since caps are a collection (or a strip) of options on each LIBOR rate.

caplets



$$\ell_t^{(k)} = \frac{1}{\Delta T_k}\left[\frac{P_t(T_{k-1})}{P_t(T_k)} - 1\right]$$

$$\frac{g_t^{(k)}}{C_t} = \mathbb{E}_t^{\mathbb{Q}^C}\left[\frac{g_{T_k}^{(k)}}{C_{T_k}}\right] = \mathbb{E}_t^{\mathbb{Q}^C}\left[\frac{(\ell_{T_{k-1}}^{(k)} - K)_+}{C_{T_k}}\right]$$

$\hookrightarrow P_t(T_k)$ $\quad\hookrightarrow P_{T_k}(T_k)$ $\quad\hookrightarrow P_{T_k}(T_k) = 1$

$\ell_t^{(k)}$ is a $\mathbb{Q}^{T_k}$ - martingale.

$$g_t^{(k)} = P_t(T_k)\, \mathbb{E}^{\mathbb{Q}^{T_k}}\left[(\ell_{T_{k-1}}^{(k)} - K)_+ \right]$$

$\exists$ $\mathcal{F}$ - adapted process $(\sigma_t)_{t\geq 0}$ s.t.

$$\frac{d\ell_t^{(k)}}{\ell_t^{(k)}} = \sigma_t^{(k)}\, dW_t^{T_k}$$

LIBOR - Market Model (LMM)

more specifically lognormal - forward rate model (LFM)

all $\sigma_t^{(k)}$ are deterministic

$$\Rightarrow \quad \ell_{T_{k-1}}^{(k)} = \ell_t^{(k)}\, e^{-\frac{1}{2}\int_t^T \sigma_s^2\, ds + \int_t^T \sigma_s\, dW_s^{T_k}}$$

$$\overset{d}{=} \ell_t^{(k)}\, e^{-\frac{1}{2}\Sigma^2 + \Sigma \cdot Z^{(k)}}$$

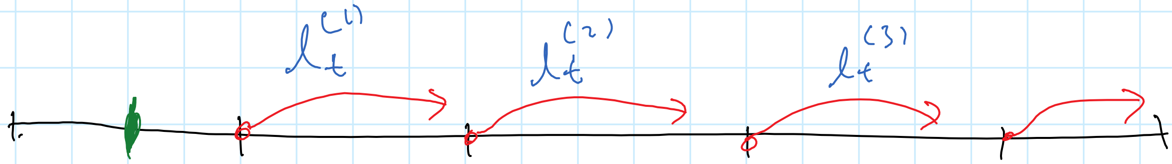
$$Z^{(k)} \underset{\mathbb{Q}^{T_k}}{\sim} \mathcal{N}(0,1), \quad \Sigma^2 = \int_t^T \sigma_s^2\, ds.$$

$$\Rightarrow \quad \boxed{g_t^{(k)} = P_t(T_k)\left[\ell_t^{(k)}\, \Phi(d_+) - K\, \Phi(d_-) \right]}$$

$$d_{\pm} = \frac{\log\left(\ell_t^{(k)}/K\right) \pm \frac{1}{2}\Sigma^2}{\Sigma}$$

LFM:
$$\frac{d\ell_t^{(k)}}{\ell_t^{(k)}} = \sigma_t^{(k)}\, dW_t^{(k)}$$

T_k



recall
$$\ell_t^{(k)} = \frac{1}{\Delta T_k}\left[\frac{P_t(T_{k-1})}{P_t(T_k)} - 1\right]$$

$$\xrightarrow{t \uparrow T_{k-1}} \frac{1}{\Delta T_k}\left[\frac{1}{P_{T_{k-1}}(T_k)} - 1\right]$$

$\ell^{(k)}$ cannot all be martingales under one measure!

$$\mathbb{Q}^{(k)} \longmapsto \mathbb{Q}^{(k-1)}$$

$$\eta_t = \left.\frac{d\mathbb{Q}^{(k-1)}}{d\mathbb{Q}^{(k)}}\right|_{\mathcal{F}_t} = \mathbb{E}_t^{\mathbb{Q}^{(k)}}\left[\frac{d\mathbb{Q}^{(k-1)}}{d\mathbb{Q}^{(k)}}\right]$$

$$= \frac{P_t(T_{k-1})\,/\,P_0(T_{k-1})}{P_t(T_k)\,/\,P_0(T_k)} = (1 + \Delta T_k\, \ell_t^{(k)}) \cdot c_0$$

$$\frac{d\eta_t}{\eta_t} = (\ ?\ )\, dW_t^{(k)}$$

$$\overline{\quad\quad}$$
$$n_t$$

$$dn_t = \Delta T_k \, d l_t^{(k)} \cdot c_0$$

$$= \Delta T_k \cdot c_0 \cdot l_t^{(k)} \, \sigma_t^{(k)} \, dW_t^{(k)}$$

$$\Rightarrow \boxed{\frac{dn_t}{n_t} = \frac{\Delta T_k \, l_t^{(k)}}{1 + \Delta T_k \, l_t^{(k)}} \, \sigma_t^{(k)} \, dW_t^{(k)}}$$

Girsanov's Thm

$$\Rightarrow \boxed{dW_t^{(k-1)} = -\frac{\Delta T_k \, l_t^{(k)}}{1 + \Delta T_k \, l_t^{(k)}} \cdot \sigma_t^{(k)} \, dt + dW_t^{(k)}}$$

$$\frac{d\, l_t^{(k)}}{l_t^{(k)}} = \sigma_t^{(k)} \, dW_t^{(k)}$$

$$\boxed{\frac{d l_t^{(k)}}{l_t^{(k)}} = \frac{\Delta T_k \, l_t^{(k)}}{1 + \Delta T_k \, l_t^{(k)}} \left(\sigma_t^{(k)}\right)^2 dt + \sigma_t^{(k)} \, dW_t^{(k-1)}}$$