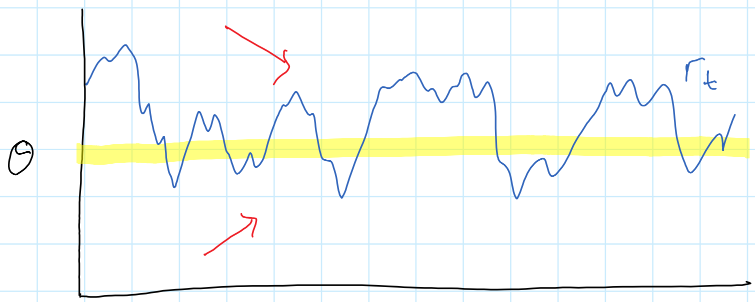
Valuing Bond Options.

In the Vasicek the short-rate process $r = (r_t)_{t \geq 0}$ satisfies the SDE:

$$\boxed{dr_t = \kappa(\theta - r_t)\, dt + \sigma\, dW_t}$$

$\mathbb{Q}$ – B.mtn

Interest rates mean-revert in this model, i.e.



κ – rate of mean-reversion

θ – level of mean-reversion

σ – volatility.

We showed that one can solve the SDE to find that:

$$\boxed{r_u - r_t = (r_t - \theta)\, e^{-\kappa(u-t)} + \theta + \sigma \int_t^u e^{-\kappa(u-s)}\, dW_s}$$

and that

$$\boxed{\int_t^T r_u\, du = \theta(T-t) - \frac{1 - e^{-\kappa(T-t)}}{\kappa}(\theta - r_t)}$$

$$\boxed{\begin{aligned}\int_t^T r_u\,du &= \theta(T-t) - \frac{1-e^{\cdots}}{\kappa}(\theta - r_t) \\ &\quad + \sigma \int_t^T \frac{1-e^{-\kappa(T-u)}}{\kappa}\,dW_u\end{aligned}}$$

Hence, since the stochastic integral has deterministic integrand, from Ito's isommetry we have

$$\boxed{\int_t^T r_u\,du \underset{\mathbb{Q}}{\sim} \mathcal{N}\left(\theta(T-t) - \frac{1-e^{-\kappa(T-t)}}{\kappa}(\theta - r_t)\;;\; \sigma^2 \int_t^T \left(\frac{1-e^{-\kappa(T-u)}}{\kappa}\right)^2 du\right)}$$

Therefore, bond prices are

$$\begin{aligned}P_t(T) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_u\,du}\right] \\ &= \exp\left\{-\theta(T-t) + \frac{1-e^{-\kappa(T-t)}}{\kappa}(\theta - r_t) + \frac{\sigma^2}{2}\int_t^T \left(\frac{1-e^{-\kappa(T-u)}}{\kappa}\right)^2 du\right\}\end{aligned}$$

which we can write in the affine form

$$\boxed{P_t(T) = \exp\{A(t;T) - B(t;T)\,r_t\}}$$

where,

$$A(t;T) = \left(\frac{1-e^{-\kappa(T-t)}}{\kappa} - (T-t)\right)\theta + \frac{\sigma^2}{2}\int_t^T \left(\frac{1-e^{-\kappa(T-u)}}{\kappa}\right)^2 du$$

and
$$B(t;T) = \frac{1-e^{-\kappa(T-t)}}{\kappa}.$$

Next to value the bond call option which pays
$$H = (P_{T_0}(T) - K)_+ \quad \text{at } T_0$$
we use the identity:
$$H = \underbrace{P_{T_0}(T)\, \mathbb{1}_{P_{T_0}(T)>K}}_{F} - \underbrace{K\, \mathbb{1}_{P_{T_0}(T)>K}}_{G}$$

and let's value each component separately which we denote by $f = (f_t)_{t\in[0,T_0]}$ & $g = (g_t)_{t\in[0,T_0]}$

① For g it is convenient to use $P(T_0)$ as a numeraire asset and we have

$$\frac{g_t}{P_t(T_0)} = \mathbb{E}_t^{\mathbb{Q}^{T_0}}\left[\frac{g_{T_0}}{\underbrace{P_{T_0}(T_0)}_{\to 1}}\right]$$

$$= \mathbb{E}_t^{\mathbb{Q}^{T_0}}\left[K\, \mathbb{1}_{P_{T_0}(T)>K}\right]$$

$$= K \cdot \mathbb{Q}_t^{T_0}\left(P_{T_0}(T) > K\right)$$

now define $X = (X_t)_{t \in [0, T_0]}$ with

$$X_t = \frac{P_t(T)}{P_t(T_0)}$$ and notice that

$$X_t \xrightarrow[t \uparrow T_0]{} \frac{P_{T_0}(T)}{P_{T_0}(T_0)} = P_{T_0}(T)$$

Hence,

$$\frac{g_t}{P_t(T_0)} = K \, \mathbb{Q}_t^{T_0}(X_{T_0} > K)$$

Now, since X is the relative price of

$P_t(T)$ to $P_t(T_0)$, then X is a
(traded asset) (numeraire)

$\mathbb{Q}^{T_0}$ – martingale, and has zero drift.

$\therefore$ from Ito's quotient rule

$$\frac{dX_t}{X_t} = \frac{dP_t(T)}{P_t(T)} - \frac{dP_t(T_0)}{P_t(T_0)}$$

$$- \frac{d[P(T_0), P(T_0)]_t}{P_t(T_0)\, P_t(T_0)} + \frac{d[P(T_0), P(T)]_t}{P_t(T_0)\, P_t(T)}$$

so we need $\dfrac{dP_t(T)}{P_t(T)}$. recall that

$$P_t(T) = \exp\left\{ A(t;T) - B(t;T)\, r_t \right\}$$

and $dr_t = \kappa(\theta - r_t)\,dt + \sigma\, dW_t$

so from Ito's lemma we have

$$dP_t(T) = (\partial_t + \mathcal{L}^r) P_t(T)\, dt + \partial_r P_t(T) \cdot \sigma\, dW_t$$

$$= \underbrace{(\partial_t + \mathcal{L}^r) P_t(T)}\, dt - \sigma B(t;T)\, P_t(T)\, dW_t$$

$r_t P_t(T)$ since $P_t(T)$ is traded and also from Feynman-Kac.

[recall $\mathcal{L}^r = \kappa(\theta - r)\partial_r + \frac{1}{2}\sigma^2 \partial_{rr}$ is the generator of r]

$$\Rightarrow \boxed{\frac{dP_t(T)}{P_t(T)} = r_t\, dt - \sigma B(t;T)\, dW_t}$$

Thus, inserting into the expression above for $\frac{dX_t}{X_t}$

$$\Rightarrow \frac{dX_t}{X_t} = \sigma\left(B(t;T_0) - B(t;T)\right) dW_t + \sigma^2\left(B(t;T_0) - B(t;T)\right) B(t;T_0)\, dt$$

but from numeraire change results $W_t^{T_0}$ defined as the solution to the SDE

$$dW_t^{T_0} = +\sigma B(t;T_0)\, dt + dW_t$$

is a $\mathbb{Q}^{T_0}$ - B-motion. and so write

$$\frac{dX_t}{X_t} = \sigma\left(B(t;T_0) - B(t;T)\right)\left(-\sigma B(t;T_0)\, dt + dW_t^{T_0}\right)$$

$$\frac{dX_t}{X_t} = \sigma\left(B(t;T_0) - B(t;T)\right)\left(-\sigma B(t;T_0)\,dt + dW_t^{T_0}\right)$$

$$+ \sigma^2\left(B(t;T_0) - B(t;T)\right) B(t;T_0)\,dt$$

$$\boxed{\frac{dX_t}{X_t} = \underbrace{\sigma\left(B(t;T_0) - B(t;T)\right)}_{u(t)} dW_t^{T_0}}$$

Note: there is no drift when we use $W_t^{T_0}$...

the measure change causes the drift to vanish.

Back to computing option price...

$$\frac{g_t}{P_t(T_0)} = K\, \mathbb{Q}_t^{T_0}\left(X_{T_0} > K\right)$$

and from

$$\frac{dX_t}{X_t} = u(t)\, dW_t^{T_0}$$

$$\Rightarrow \quad X_{T_0} = X_t \exp\left\{-\frac{1}{2}\int_t^T u^2(s)\,ds + \int_t^T u(s)\,dW_s^{T_0}\right\}$$

$$\overset{d}{=} X_t \exp\left\{-\frac{1}{2}\Sigma^2 + \Sigma \cdot Z_1^{T_0}\right\}$$

where $\Sigma^2 = \int_t^T u^2(s)\,ds$

and $Z_1^{T_0} \underset{\mathbb{Q}^{T_0}}{\sim} \mathcal{N}(0,1)$

$$\Rightarrow \quad g_t = K\, P_t(T_0) \cdot \mathbb{Q}^{T_0}\left(X_t\, e^{-\frac{1}{2}\Sigma^2 + \Sigma \cdot Z_1^{T_0}} > K\right)$$

$$= K\, P_t(T_0) \cdot \mathbb{Q}^{T_0}\left(Z_1^{T_0} > \frac{\log(K/X_t) + \frac{1}{2}\Sigma^2}{\Sigma} \right)$$

$$= K\, P_t(T_0)\, \Phi\left(\frac{\log(X_t/K) - \frac{1}{2}\Sigma^2}{\Sigma} \right)$$

std. normal cdf

$$= K\, P_t(T_0)\, \Phi\left(\underbrace{\frac{\log\left(\frac{P_t(T)}{P_t(T_0)K}\right) - \frac{1}{2}\Sigma^2}{\Sigma}}_{d_-} \right)$$

Now turning to f, which as $f_{T_0} = P_{T_0}(T)\, \mathbb{1}_{P_{T_0}(T) > K}$

we use $P(T)$ as numeraire now.

$$\Rightarrow \frac{f_t}{P_t(T)} = \mathbb{E}^{\mathbb{Q}^T}\left[\frac{f_{T_0}}{P_{T_0}(T)} \right]$$

$$= \mathbb{E}^{\mathbb{Q}^T}\left[\frac{\cancel{P_{T_0}(T)}\, \mathbb{1}_{P_{T_0}(T) > K}}{\cancel{P_{T_0}(T)}} \right]$$

$$= \mathbb{Q}^T\left(P_{T_0}(T) > K \right)$$

$$= \mathbb{Q}^T\left(X_{T_0} > K \right)$$

So we need to write X in terms of $\mathbb{Q}^T$- B.mtn.

recall that

$$\frac{d\mathbb{Q}^T}{d\mathbb{Q}^{T_0}} = \frac{P_T(T) \,/\, P_0(T)}{Y_T \,/\, Y_0} \qquad \text{and}$$

$$Y_t = \begin{cases} P_t(T_0), & t < T_0 \\ e^{\int_{T_0}^t r_s\,ds}, & t \geq T_0 \end{cases}$$

and

$$\eta_t \triangleq \mathbb{E}_t^{\mathbb{Q}^{T_0}}\left[\frac{d\mathbb{Q}^T}{d\mathbb{Q}^{T_0}}\right] = \frac{P_t(T) \,/\, P_0(T)}{Y_t \,/\, Y_0}$$

since $\frac{P_t(T)}{Y_t}$ is a $\mathbb{Q}^{T_0}$-mtg.

$$\Rightarrow \quad \frac{d\eta_t}{\eta_t} = (\text{vol of } P_t(T) - \text{vol of } Y_t)$$

$$= \left(-\sigma\, B(t;T)\right) dW_t^{T_0} - \left(-\sigma\, B(t;T_0)\, \mathbb{1}_{t<T_0}\right) dW_t^{T_0}$$

$$= \sigma\left(B(t;T_0)\,\mathbb{1}_{t<T_0} - B(t;T)\right) dW_t^{T_0}$$

Hence, from Girsanov's Theorem: W_t^T which satisfies:

$$dW_t^T = -\underbrace{\sigma\left(B(t;T_0)\,\mathbb{1}_{t<T_0} - B(t;T)\right)}_{\text{same as } u(t) \text{ from above on } t\in[0,T_0)} dt + dW_t^{T_0}$$

is a $\mathbb{Q}^T$-martingale.

Now from before we had that

$$\frac{dX_t}{X_t} = u(t)\, dW_t^{T_0}$$

$$= u(t)\left(dW_t^T + u(t)\,dt\right), \quad t\in[0,T_0)$$

$$= u^2(t)\,dt + u(t)\,dW_t^T$$

$$\Rightarrow \quad X_{T_0} = X_t \exp\left\{ \int_t^{T_0} \left(u^2(s) - \tfrac{1}{2} u^2(s) \right) ds + \int_t^{T_0} u(s)\, dW_s^T \right\}$$

$$\stackrel{d}{=} X_t \exp\left\{ + \tfrac{1}{2} \Sigma^2 + \Sigma \cdot Z_1^T \right\}$$

where $Z_1^T \underset{\mathbb{Q}^T}{\sim} \mathcal{N}(0,1)$

Hence we have,

$$f_t = P_t(T) \cdot \Phi\Bigg(\underbrace{\frac{\log \frac{P_t(T)}{K P_t(T_0)} + \frac{1}{2}\Sigma^2}{\Sigma}}_{d_+} \Bigg)$$

all together

$$h_t = f_t - g_t$$

$$= P_t(T)\, \mathbb{Q}^T(P_{T_0}(T) > K) - K P_t(T_0)\, \mathbb{Q}^{T_0}(P_{T_0}(T) > K)$$

$$\boxed{h_t = P_t(T)\, \Phi(d_+) - K P_t(T_0)\, \Phi(d_-)}$$

Method 2

We want to value a contract paying

$$H = (P_{T_0}(T) - K)_+ \quad \text{at} \quad T_0$$

Using T_0-bond as numeraire we have

$$\frac{h_t}{P_t(T_0)} = \mathbb{E}_t^{\mathbb{Q}^{T_0}}\left[\frac{(P_{T_0}(T) - K)_+}{\underbrace{P_{T_0}(T_0)}_{\to 1}} \right]$$

$$= \mathbb{E}_t^{\mathbb{Q}^{T_0}}\left[(X_{T_0} - K)_+ \right]$$

where as before $X_t = \frac{P_t(T)}{P_t(T_0)}$ and is a

$\mathbb{Q}^{T_0}$ - martingale.

now as in the method 1 we have that

$$\frac{dX_t}{X_t} = \underbrace{\sigma (B(t;T_0) - B(t;T))}_{u(t)} \; dW_t^{T_0}$$

and so,

$$X_{T_0} = X_t \, e^{-\frac{1}{2}\int_t^T u^2(s)\,ds + \int_t^T u(s)\,dW_s}$$

$$\overset{d}{=} X_t \, e^{-\frac{1}{2}\Sigma^2 + \Sigma \cdot Z^{T_0}}$$

and $\Sigma^2 = \int_t^T u^2(s)\, ds$

$$Z^{T_0} \underset{\mathbb{Q}^{T_0}}{\sim} \mathcal{N}(0,1)$$

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recall that in B-S

$$\frac{dS_t}{S_t} = r\,dt + \sigma\, dW_t$$

and $$S_T = S_t\, e^{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)}$$

$$\overset{d}{=} S_t\, e^{r(T-t) - \frac{1}{2}\Sigma^2 + \Sigma \cdot Z}$$

$$Z \underset{\mathbb{Q}}{\sim} \mathcal{N}(0,1) \quad \text{and } \Sigma = \sigma^2 (T-t)$$

Moreover,

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} (S_T - K)_+ \right]$$

$$= S_t\, \Phi(d_+) - K e^{-r(T-t)} \Phi(d_-)$$

$$d_\pm = \frac{\log(S_t/K) + r(T-t) \pm \frac{1}{2}\Sigma^2}{\Sigma}$$

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so apply B-S result with $r=0$ and

$$\Sigma^2 = \int_t^T u^2(s)\, ds$$

$$\Rightarrow \quad h_t = P_t(T_0)\left[X_t \, \Phi(d_+) - K \, \Phi(d_-) \right]$$

$$d_\pm = \frac{\log(X_t/K) \pm \frac{1}{2}\Sigma^2}{\Sigma}$$