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$$\frac{dX_t}{X_t} = \nu(t)\, dW_t^{T_0}$$

$$X_t \triangleq P_t(T) / P_t(T_0)$$

$$\frac{F_t}{P_t(T_0)} = \mathbb{E}_t^{\mathbb{Q}^{T_0}}\left[X_{T_0} \mathbb{1}_{X_{T_0} > K} \right]$$

$$= \mathbb{E}_t^{\mathbb{Q}^{T_0}}\left[\frac{P_{T_0}(T)}{\underbrace{P_{T_0}(T_0)}_{1}} \mathbb{1}_{\frac{P_{T_0}(T)}{\underbrace{P_{T_0}(T_0)}_{1}} > K} \right]$$

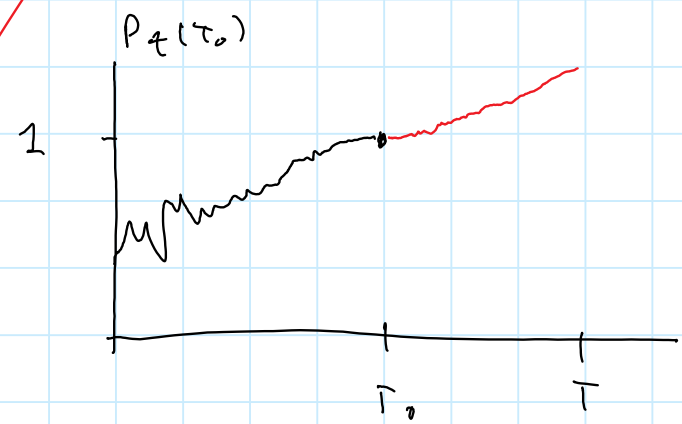
$$= \mathbb{E}_t^{\mathbb{Q}^{T_0}}\left[P_{T_0}(T)\, \mathbb{1}_{P_{T_0}(T) > K} \right]$$

$$\frac{F_t}{P_t(T)} = \mathbb{E}_t^{\mathbb{Q}^{T}}\left[\frac{\cancel{P_{T_0}(T)}\, \mathbb{1}_{P_{T_0}(T) > K}}{\cancel{P_{T_0}(T)}} \right]$$

$$= \mathbb{Q}^T\left(P_{T_0}(T) > K \right)$$

$$\frac{d\mathbb{Q}^T}{d\mathbb{Q}^{T_0}} = \frac{P_T(T) / P_0(T)}{P_T(T_0)\quad P_0(T_0)}$$

$$d\,\mathbb{Q}^{T_0} \qquad P_T(T_0) \quad P_0(T_0)$$



$$Y_t = \begin{cases} P_t(T_0), & t < T_0 \\ e^{\int_{T_0}^{t} r_s\,ds}, & t \ge T_0 \end{cases}$$

$$\frac{d\,\mathbb{Q}^{T}}{d\,\mathbb{Q}^{T_0}} = \frac{P_T(T) \,/\, P_0(T)}{Y_T \,/\, Y_0}$$

$$\eta_t \triangleq \mathbb{E}_t^{\mathbb{Q}^{T_0}}\left[\frac{d\,\mathbb{Q}^{T}}{d\,\mathbb{Q}^{T_0}} \right] \quad \text{is a } \mathbb{Q}^{T_0}\text{-mtg.}$$

$$= \frac{P_t(T) \,/\, P_0(T)}{Y_t \,/\, Y_0}$$

recall $P_t(T) = e^{A(t;T) - B(t;T)\, r_t}$

$$d r_t = \kappa(\theta - r_t)\,dt + \sigma\, dW_t^{\mathbb{B}}$$

$$d\eta_t = 0\,dt - B(t;T)\,\sigma\, dW_t^{T_0}$$

$$\frac{d\eta_t}{\eta_t} = 0\,dt - B(t;T)\,\sigma\,dW_t^{T_0}$$

$$\qquad + B(t;T_0)\,\sigma\,\mathbb{1}_{t<T_0}\,dW_t^{T_0}$$

$$= \sigma\left(B(t;T_0)\,\mathbb{1}_{t<T_0} - B(t;T)\right)dW_t^{T_0}$$

$$dW_t^{T} = -\sigma\left(B(t;T_0)\,\mathbb{1}_{t<T_0} - B(t;T)\right)dt$$

$$\qquad + dW_t^{T_0}$$

$$\frac{F_t}{P_t(T)} = \mathbb{Q}^T\left(P_{T_0}(T) > K\right)$$

but $X_t = \dfrac{P_t(T)}{P_t(T_0)} \xrightarrow[t \uparrow T_0]{} P_{T_0}(T)$

$$\frac{dX_t}{X_t} = \overbrace{\sigma\left(B(t;T_0) - B(t;T)\right)}^{u(t)}\,dW_t^{T_0}, \qquad t \le T_0$$

$$= u(t)\,dW_t^{T_0}$$

$$= u(t)\left(dW_t^{T} + u(t)\,dt\right)$$

$$= u^2(t)\,dt + u(t)\,dW_t^{T}$$

$= u(t)\, dt + u(t)\, dW_t$

$$\Rightarrow \quad X_{T_0} = X_t \cdot e^{+\frac{1}{2}\int_t^{T_0} u^2(s)\, ds + \int_t^{T_0} u(s)\, dW_s^T}$$

NB: $$\int_t^{T_0} u(s)\, dW_s^T \sim \mathcal{N}\Big(0; \underbrace{\int_t^{T_0} u^2(s)\, ds}_{\sigma^2}\Big)$$

$$\Rightarrow \quad F_t = P_t(T)\, \mathbb{Q}^T\big(P_{T_0}(T) > K\big)$$

$$= P_t(T)\, \mathbb{Q}^T\big(X_{T_0} > K\big)$$

$$= P_t(T)\, \mathbb{Q}^T\big(X_t\, e^{-\cdots} > K\big)$$

$$= \ldots$$

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$$H = C P_{T_0}(T) - K)_+$$

$$\boxed{\begin{aligned} V_t = \; & P_t(T)\, \mathbb{Q}^T\left(P_{T_0}(T) > K\right) \\ & - K\, P_t(T_0)\, \mathbb{Q}^{T_0}\left(P_{T_0}(T) > K\right) \end{aligned}}$$

$$P_t(T) = \mathbb{E}^{\mathbb{Q}^B}\left[e^{-\int_t^T r_u\,du} \right]$$

$$= e^{A(t;T) - B(t;T) r_t}$$

$$dr_t = \kappa(\theta - r_t)\,dt + \sigma\, dW_t^B$$

$$g_t = e^{\kappa t} r_t$$

$$dg_t = d(e^{\kappa t})\, r_t + e^{\kappa t}\, dr_t + \underbrace{d[e^{\kappa t}, r_t]}_{\to 0}$$

$$= \kappa\, e^{\kappa t} r_t\, dt$$

$$+ e^{\kappa t}\left(\kappa(\theta - r_t)\,dt + \sigma\, dW_t^B \right)$$

$$= e^{\kappa t} \kappa \theta \, dt + \sigma e^{\kappa t} \, dW_t^{B}$$

$$g_T - g_t = \kappa \theta \int_t^T e^{\kappa u} \, du + \sigma \int_t^T e^{\kappa u} \, dW_u^{B}$$

$$\Rightarrow g_T = g_t + (e^{\kappa T} - e^{\kappa t}) \theta + \sigma \int_t^T e^{\kappa u} \, dW_u^{B}$$

$$e^{\kappa T} r_T = e^{\kappa t} r_t + (e^{\kappa T} - e^{\kappa t}) \theta + \sigma \int_t^T e^{\kappa u} \, dW_u^{B}$$

$$\Rightarrow \boxed{r_T = e^{-\kappa(T-t)} r_t + (1 - e^{-\kappa(T-t)}) \theta + \sigma \int_t^T e^{-\kappa(T-u)} \, dW_u^{B}}$$

$$dr_t = \kappa(\theta - r_t) dt + \sigma \, dW_t^{B}$$

$$\Rightarrow r_T - r_t = \kappa \theta (T-t) - \kappa \int_t^T r_u \, du + \sigma \int_t^T dW_t^{B}$$

$$\Rightarrow \int_t^T r_u \, du = \theta (T-t) + \frac{\sigma}{\kappa} \int_t^T dW_t^{B}$$

$$- \frac{r_T - r_t}{\kappa}$$

$$= \theta(T-t) - \frac{\left(1 - e^{-\kappa(T-t)}\right)}{\kappa}\left(\theta - r_t\right)$$

$$- \frac{\sigma}{\kappa}\int_t^T e^{-\kappa(T-u)}\, dW_u^B$$

$$+ \frac{\sigma}{\kappa}\int_t^T dW_u^B$$

$$\Rightarrow \boxed{\int_t^T r_u\, du = \overbrace{\theta(T-t) - \frac{1 - e^{-\kappa(T-t)}}{\kappa}\left(\theta - r_t\right)}^{\mu} + \frac{\sigma}{\kappa}\int_t^T \left(1 - e^{-\kappa(T-u)}\right) dW_u^B}$$

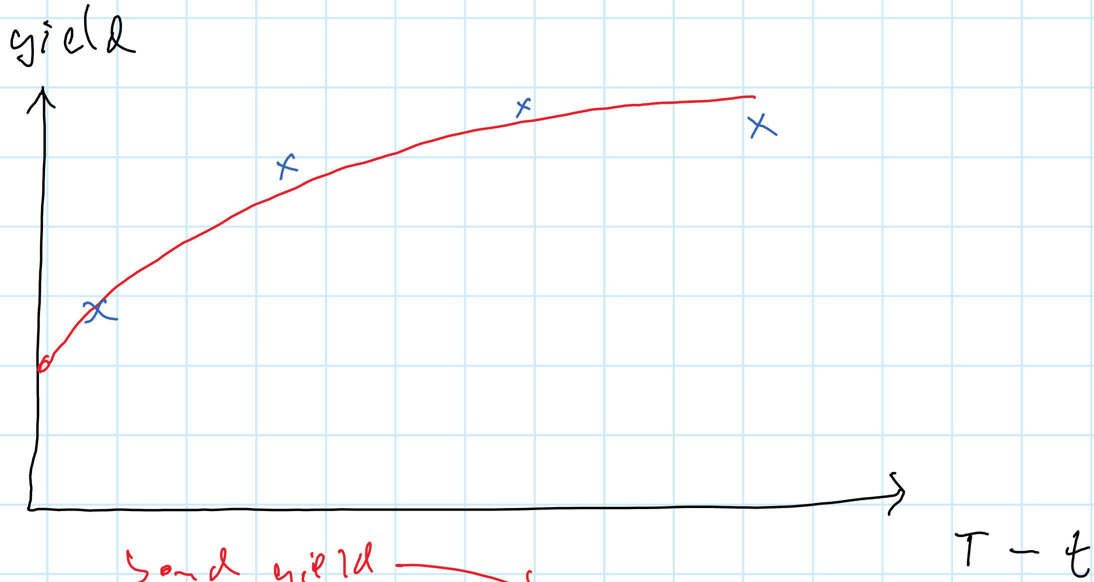
$$\int_t^T r_u\, du \Big|_{\mathcal{F}_t} \sim \mathcal{N}\left(\mu(t,T;r_t)\,;\, \zeta^2(t,T)\right)$$

$$\zeta^2(t,T) = \frac{\sigma^2}{\kappa^2}\int_t^T \left(1 - e^{-\kappa(T-u)}\right)^2 du$$

$$P_t(T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_u\, du}\right]$$

$$= \exp\left\{ -\mu + \frac{1}{2}\zeta^2 \right\}$$

$$= \exp\left\{ -\theta(T-t) + \frac{1-e^{-\kappa(T-t)}}{\kappa}\,\theta + \frac{\sigma^2}{2\kappa^2}\int_t^T \left(1-e^{-\kappa(T-u)}\right)^2 du - \frac{1-e^{-\kappa(T-t)}}{\kappa}\, r_t \right\}$$



bond yield

$$P_t(T) = e^{-y_t(T)\,(T-t)}$$

$$= e^{-\int_t^T f_t(u)\,du}$$

instaneous forward rate

$$= \mathbb{E}^{\mathbb{Q}^B}\left[e^{-\int_t^T r_u\,du} \right]$$

short rate

short rate.

$$dx_t = -\kappa x_t \, dt + \sigma \, dW_t^{\mathbb{B}}$$

$$r_t = \theta_t + x_t$$

deterministic

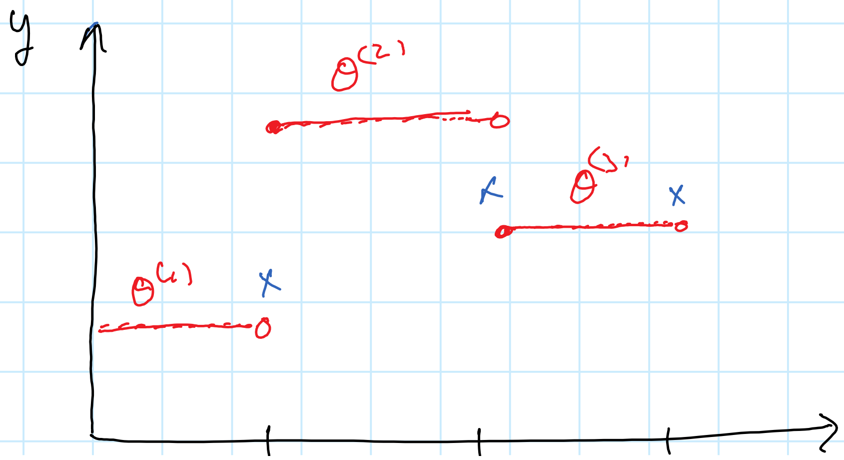
Hull - White

Extended Vasicek

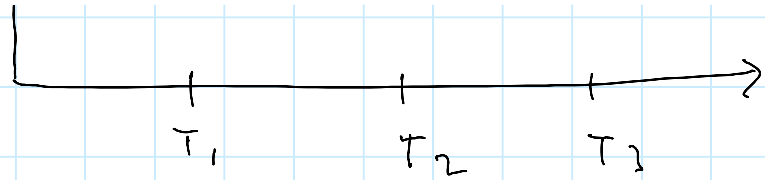
$$\Rightarrow \quad P_t(T) = \mathbb{E}^{\mathbb{Q}^{\mathbb{B}}}\left[e^{-\int_t^T (\theta_u + x_u)\, du} \right]$$

$$= e^{-\int_t^T \theta_u \, du} \cdot \mathbb{E}^{\mathbb{Q}^{\mathbb{B}}}\left[e^{-\int_t^T x_u \, du} \right]$$

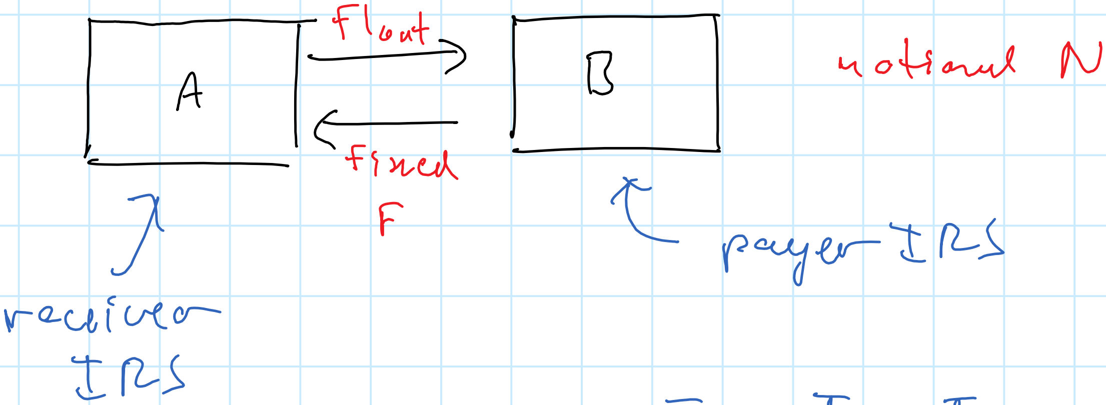
$$P_t(T) = \exp\left\{ -\int_t^T \theta_u \, du - \frac{1-e^{-\kappa(T-t)}}{\kappa} \underbrace{x_t}_{\to\, r_t - \theta_t} + \frac{\sigma^2}{2\kappa^2} \int_t^T \left(1 - e^{-\kappa(T-u)}\right)^2 du \right\}$$



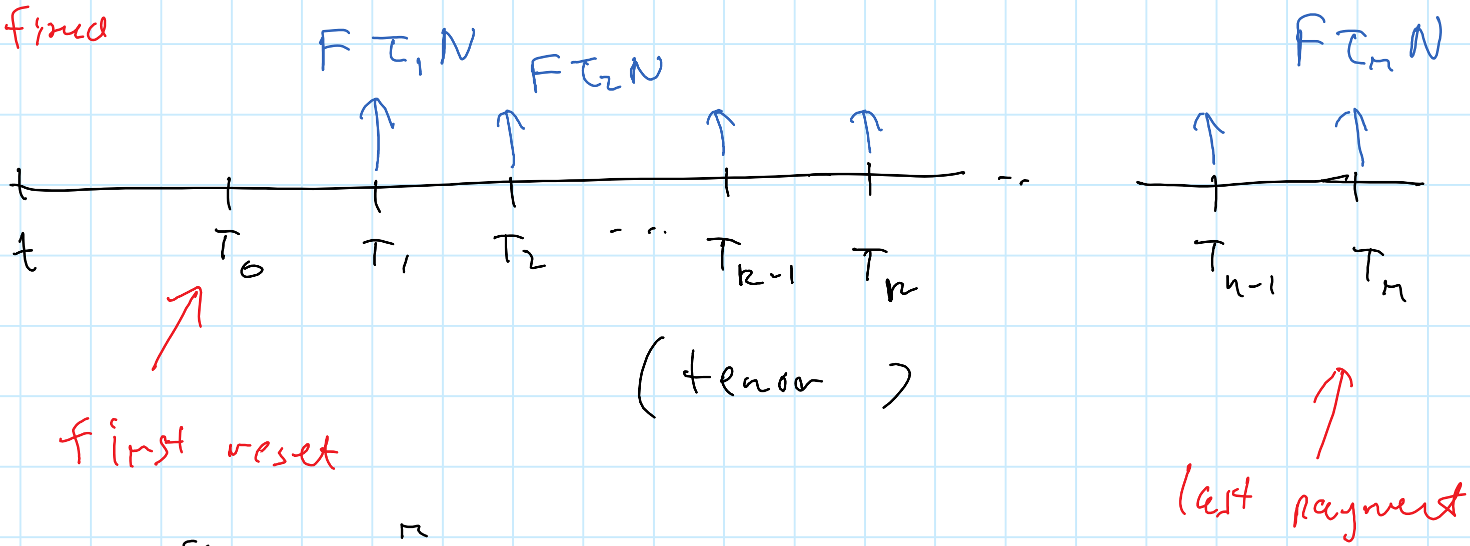
bootstrapping


T_1
T_2
T_3

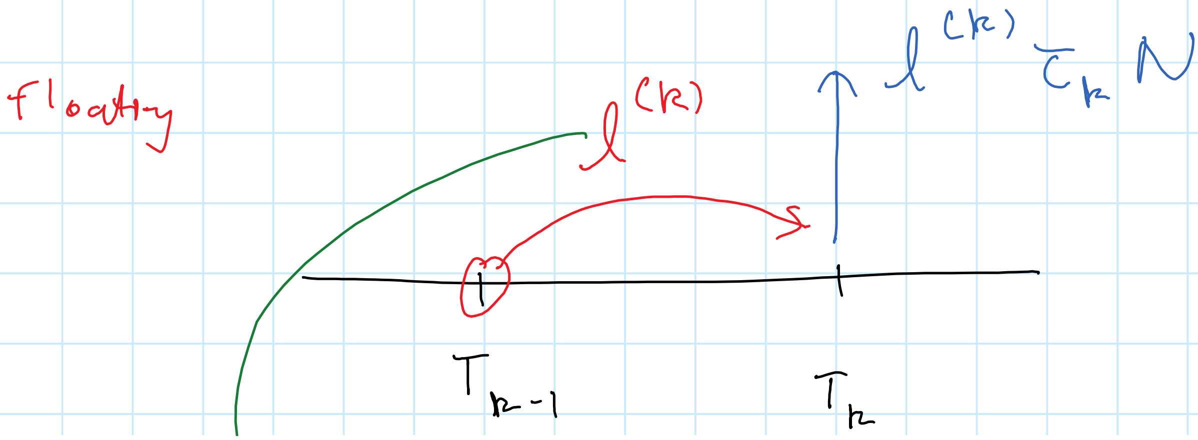
Interest Rate Swaps (IRS)



$$\tau_k = T_k - T_{k-1}$$



$$V_t^{fix} = \sum_{k=1}^{n} F N \tau_k P_t(T_k)$$



T_{k-1} $\qquad\qquad$ T_k

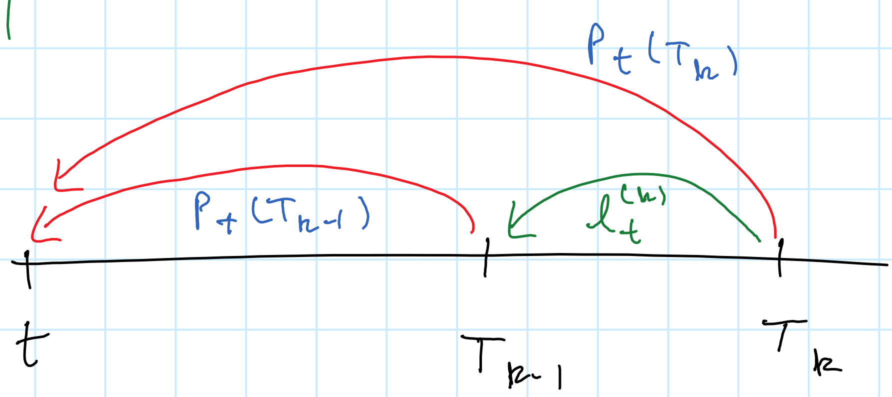
LIBOR rate
(London Interbank offer rate)

$$P_{T_{k-1}}(T_k) \triangleq \left(1 + \ell^{(k)} \tau_k\right)^{-1}$$

$$\Rightarrow \quad \ell^{(k)} = \frac{1}{\tau_k}\left[\frac{1}{P_{T_{k-1}}(T_k)} - 1\right]$$

Introduce

$$\ell_t^{(k)} = \frac{1}{\tau_k}\left[\frac{P_t(T_{k-1})}{P_t(T_k)} - 1\right] \xrightarrow[t \uparrow T_{k-1}]{} \ell^{(k)}$$



$$P_t(T_k) = P_t(T_{k-1})\left(1 + \ell_t^{(k)} \tau_k\right)^{-1}$$

$V_{T_k}^{fl,k}$

fl,k $\qquad$ $\mathbb{Q}^B\Big[$ $\qquad$ $\ell^{(k)}$

$$V_{T_k}^{fl,k}$$

$$\frac{V_t^{fl,k}}{B_t} = \mathbb{E}_t^{\mathbb{Q}^B}\left[\frac{\tau_k\, \ell_{T_{k-1}}^{(k)}\, N}{B_{T_k}}\right]$$

$$\frac{V_t^{fl,k}}{P_t(T_k)} = \mathbb{E}_t^{\mathbb{Q}^{T_k}}\left[\frac{\tau_k\, \ell_{T_{k-1}}^{(k)}\, N}{P_{T_k}(T_k)}\right]$$

1

Note $\ell_t^{(k)}$ is a $\mathbb{Q}^{T_k}$-mtg-

(so $\mathbb{E}_t^{\mathbb{Q}^{T_k}}[\ell_{T_{k-1}}^{(k)}] = \ell_t^{(k)}$)

$$\Rightarrow V^{fl,k} = P_t(T_k)\, \tau_k\, \ell_t^{(k)}\, N$$

$$= \left(P_t(T_{k-1}) - P_t(T_k)\right) N$$

$$\Rightarrow V^{fl} = \sum_{k=1}^{n} \left(P_t(T_{k-1}) - P_t(T_k)\right) N$$

$$\boxed{V_t^{fl} = \left(P_t(T_0) - P_t(T_n)\right) N}$$

swap-rate S_t is s.t. such that

swap-rate S_t is such that

$$V_t^{fixed} = V_t^{fl}$$

$$\boxed{S_t = \frac{P_t(T_0) - P_t(T_n)}{\sum_{k=1}^{n} P_t(T_k)\,\tau_k}}$$