Wednesday, November 11, 2015 2:09 PM

no arb $\therefore$ $\exists\ \mathbb{Q}^B \sim \mathbb{P}$ s.t. $\forall$ all traded assets A we have

$$\frac{A_t}{B_t} = \mathbb{E}_t^{\mathbb{Q}^B}\left[\frac{A_u}{B_u} \right], \quad u > t$$

$$\left(\mathbb{E}^{\mathbb{Q}^B}[\ \cdot \mid \mathcal{F}_t] \right)$$

$B_t > 0$ a.s. $\forall t$.

Further suppose $\exists\ C_t > 0$ a.s. $\forall t$
then we must also have that $\exists\ \mathbb{Q}^C \sim \mathbb{P}$ s.t.
$\forall$ traded assets A we have

$$\frac{A_t}{C_t} = \mathbb{E}_t^{\mathbb{Q}^C}\left[\frac{A_u}{C_u} \right], \quad u > t.$$

Since $\mathbb{Q}^B \sim \mathbb{P}$ and $\mathbb{Q}^C \sim \mathbb{P}$ then $\mathbb{Q}^B \sim \mathbb{Q}^C$

$$\mathbb{E}^{\mathbb{Q}^C}[\mathbb{1}_A] = \mathbb{E}^{\mathbb{Q}^B}\left[\mathbb{1}_A \cdot \frac{d\mathbb{Q}^C}{d\mathbb{Q}^B} \right]$$

$$\parallel$$

$$\mathbb{Q}^C(A) = \left(\frac{\mathbb{Q}^C(A)}{\mathbb{Q}^B(A)} \right) \cdot \mathbb{Q}^B(A)$$

$$\frac{\mathbb{Q}^C(\omega)}{\mathbb{Q}^B(\omega)} : \Omega \mapsto \mathbb{R}$$

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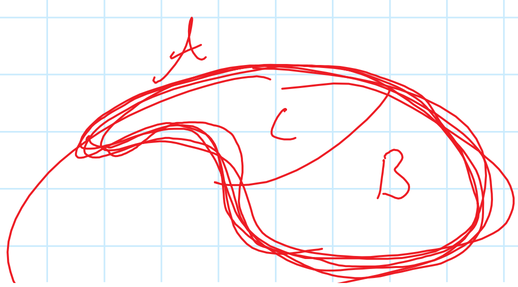


$$\mathbb{E}_t^{\mathbb{Q}^C}\left[\mathbb{1}_A \right] = \frac{\mathbb{E}_t^{\mathbb{Q}^B}\left[\mathbb{1}_A \cdot \frac{d\mathbb{Q}^C}{d\mathbb{Q}^B} \right]}{\mathbb{E}_t^{\mathbb{Q}^B}\left[\frac{d\mathbb{Q}^C}{d\mathbb{Q}^B} \right]}$$

$$A_t = \mathbb{E}_t^{\mathbb{Q}^B}\left[\frac{A_u}{B_u} \right] B_t$$

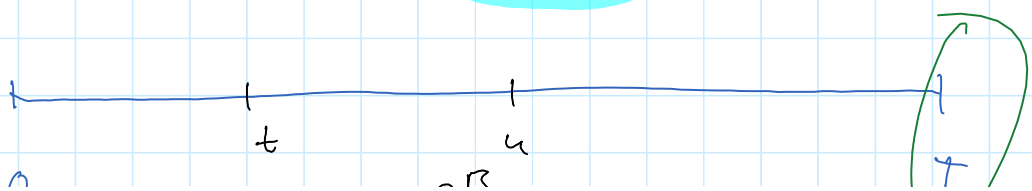
also

$$A_t = \mathbb{E}_t^{\mathbb{Q}^C}\left[\frac{A_u}{C_u} \right] C_t$$

$$\therefore \quad \mathbb{E}_t^{\mathbb{Q}^B}\left[\frac{A_u}{B_u} \right] \cdot B_t = \mathbb{E}_t^{\mathbb{Q}^C}\left[\frac{A_u}{C_u} \right] \cdot C_t$$

$$\Rightarrow \quad \mathbb{E}_t^{\mathbb{Q}^C}\left[\frac{A_u}{C_u} \right] = \frac{\mathbb{E}_t^{\mathbb{Q}^B}\left[\frac{A_u}{B_u} \right]}{C_t / B_t}$$

$$= \frac{\mathbb{E}_t^{\mathbb{Q}^B}\left[\frac{A_u}{C_u} \cdot \frac{C_u}{B_u} \right]}{C_t / B_t}$$



$0 \qquad t \qquad u \qquad T$

$$\frac{C_u}{B_u} = \mathbb{E}_u^{\mathbb{Q}^B}\left[\frac{C_T}{B_T}\right]$$

also $$\frac{C_t}{B_t} = \mathbb{E}_t^{\mathbb{Q}^B}\left[\frac{C_T}{B_T}\right]$$

$$\frac{C_T}{B_T} \rightarrow ? \quad \frac{d\mathbb{Q}^C}{d\mathbb{Q}^B}$$

$$\therefore \quad \mathbb{E}_t^{\mathbb{Q}^C}\left[\frac{A_u}{C_u}\right] = \frac{\mathbb{E}_t^{\mathbb{Q}^B}\left[\frac{A_u}{C_u}\cdot \mathbb{E}_u^{\mathbb{Q}^B}\left[\frac{C_T}{B_T}\right]\right]}{\mathbb{E}_t^{\mathbb{Q}^B}\left[\frac{C_T}{B_T}\right]}$$

$$= \frac{\mathbb{E}_t^{\mathbb{Q}^B}\left[\mathbb{E}_u^{\mathbb{Q}^B}\left[\frac{A_u}{C_u}\cdot\frac{C_T}{B_T}\right]\right]}{\mathbb{E}_t^{\mathbb{Q}^B}\left[\frac{C_T}{B_T}\right]}$$

$$= \frac{\mathbb{E}_t^{\mathbb{Q}^B}\left[\frac{A_u}{C_u}\cdot\frac{C_T}{B_T}\right]}{\mathbb{E}_t^{\mathbb{Q}^B}\left[\frac{C_T}{B_T}\right]}$$

clearly $\frac{C_T}{B_T} \geq 0$ a.s.

and $$\mathbb{E}^{\mathbb{Q}^B}\left[\frac{C_T}{B_T}\right] = \frac{C_0}{B_0}$$

hence $$\boxed{\frac{d\mathbb{Q}^C}{d\mathbb{Q}^B} = \frac{C_T / C_0}{B_T / B_0}}$$ is

the Radon-Nikodym derivative we seek

seek.

(b/c R-N derivatives have

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{d\mathbb{Q}^*}{d\mathbb{Q}}\right] = \mathbb{E}^{\mathbb{Q}^*}[1] = 1)$$

we know that $\dfrac{d\mathbb{Q}^C}{d\mathbb{Q}^B} = \dfrac{C_T / C_0}{B_T / B_0}$

let's assume that

$$dB_t = \mu_t^B \, B_t \, dt + \sigma_t^B \, B_t \, dW_t^{B,1} \quad (1)$$

$$dC_t = \mu_t^C \, C_t \, dt + \sigma_t^C \, C_t \, dW_t^{B,2} \quad (2)$$

$W^{B,k} = (W_t^{B,k})_{t \geq 0}$ are correlated $\mathbb{Q}^B$ - B.mtn

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$$(1) \Rightarrow B_t = B_0 \exp\left\{ \int_0^t \left(\mu_u^B - \tfrac{1}{2}(\sigma_u^B)^2\right) du + \int_0^t \sigma_u^B \, dW_u^{B,1} \right\}$$

$$(2) \Rightarrow C_t = C_0 \exp\left\{ \int_0^t \left(\mu_u^C - \tfrac{1}{2}(\sigma_u^C)^2\right) du + \int_0^t \sigma_u^C \, dW_u^{B,2} \right\}$$

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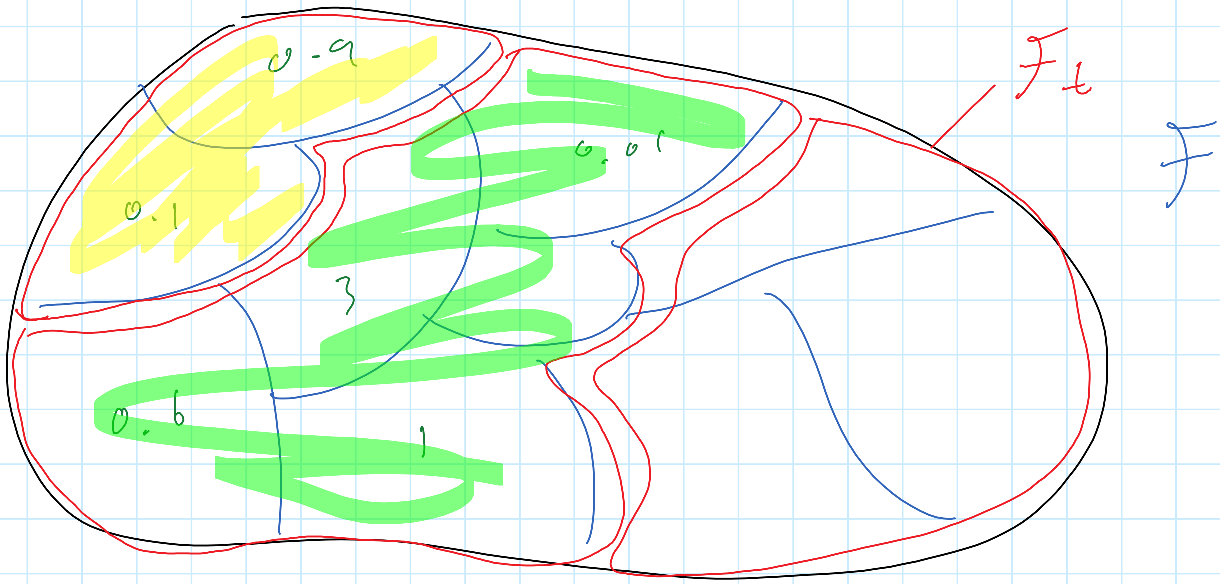
note: $\dfrac{d\mathbb{Q}^C}{d\mathbb{Q}^B} \in L^1(\mathbb{Q}^B)$

define $\eta_t \triangleq \mathbb{E}_t^{\mathbb{Q}^B}\left[\dfrac{d\mathbb{Q}^C}{d\mathbb{Q}^B} \right]$ $\quad$ RDD-mtg.



$$\eta_t = \mathbb{E}_t^{\mathbb{Q}^B}\left[\frac{C_T / C_0}{B_T / B_0} \right] = \frac{C_t / C_0}{B_t / B_0}$$

is a $\mathbb{Q}^B$ - mtg by the fundamental theorem of finance.

$$Z = \mathcal{E}\left(\int_0^T \lambda_s \, dW_s^B \right)$$

$$\frac{dz_t}{z_t} = \lambda_t \, dW_t^B \quad , \quad Z = z_T$$

Girsanov's Theorem $\Rightarrow$ $W_t^Z = -\int_0^t \lambda_s \, ds + W_t^B$
is a $\mathbb{Q}^Z$ - Brownian.

$W^{B,1} \perp W^{B,2}$

$$Z = \mathcal{E}\left(\int_0^T \lambda_s^1 \, dW_s^{B,1} \right) \mathcal{E}\left(\int_0^T \lambda_s^2 \, dW_s^{B,2} \right)$$

$$\Rightarrow \quad W_t^{Z,1} = - \int_0^t \lambda_s^1 \, ds + W_t^{B,1}$$

$$W_t^{Z,2} = - \int_0^t \lambda_s^2 \, ds + W_t^{B,2}$$

$$W_t^{Z,2} = - \int_0^t \lambda_s^2 \, ds + W_t^{B,2}$$

are two ind. $\mathbb{Q}^Z$ - B. mtn.

suppose I have a third B. mtn $W^{B,3}$ &

$$[W^{B,3}, W^{B,1}]_t = \rho t$$

$$W^{B,3} = \rho W^{B,1} + \sqrt{1-\rho^2} \, W^{B,2}$$

suppose $\lambda^2 = 0$.

$$W^{Z,3} = \rho W^{Z,1} + \sqrt{1-\rho^2} \, W^{Z,2}$$

$$= - \int_0^t \rho \lambda_s^1 \, ds + \underbrace{\rho W^{B,1} + \sqrt{1-\rho^2} \, W^{B,2}}_{W^{B,3}}$$

$$\boxed{W_t^{Z,3} = - \int_0^t \rho \lambda_s^1 \, ds + W_t^{B,3}}$$

so since η_t is a $\mathbb{Q}^B$-mtg we must have

$$d\eta_t = (\quad) \, dW_t^{B,1} + (\quad) \, dW_t^{B,2} + 0 \, dt$$

$$\frac{d(C_t/B_t)}{(C_t/B_t)} = \frac{dC_t}{C_t} - \frac{dB_t}{B_t} + \frac{d[B,B]_t}{B_t^2} - \frac{d[C,B]_t}{C_t B_t}$$

$$\frac{d(C_t/B_t)}{(C_t/B_t)} = \frac{dC_t}{C_t} - \frac{dB_t}{B_t} + \frac{d[B,B]_t}{B_t^2} - \frac{d[C,B]_t}{C_t B_t}$$

$$= (\mu_t^C\, dt + \sigma_t^C\, dW_t^{B,2})$$

$$- (\mu_t^B\, dt + \sigma_t^B\, dW_t^{B,1})$$

$$+ (\sigma_t^B)^2\, dt - \rho\, \sigma_t^B \sigma_t^C\, dt$$

$$= (\;0\;)\, dt + \sigma_t^C\, dW_t^{B,2} - \sigma_t^B\, dW_t^{B,1}$$

$$\Rightarrow \boxed{\frac{d\eta_t}{\eta_t} = \sigma_t^C\, dW_t^{B,2} - \sigma_t^B\, dW_t^{B,1}}$$

if B was the bank account then

$\sigma_t^B = 0$ &

$$dW_t^{C,1} = -\rho\, \sigma_t^C\, dt + dW_t^{B,1}$$

$$dW_t^{C,2} = -\sigma_t^C\, dt + dW_t^{B,2}$$

suppose B is the bank account (const r)
S is an asset price (GBM)

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t$$

$$\Rightarrow S_T = S_0\, e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$$

value a put option

$$\frac{V_0}{B_0} = \mathbb{E}_0^{Q^B}\left[\frac{(K - S_T)_+}{B_T} \right] \quad (1)$$

but $(K - S_T)_+$

$$= \underbrace{K\, \mathbb{1}_{S_T < K}}_{F} - \underbrace{S_T\, \mathbb{1}_{S_T < K}}_{G}$$

$$\frac{F_0}{B_0} = \mathbb{E}^{Q^B}\left[\frac{K\, \mathbb{1}_{S_T<K}}{B_T} \right]$$

$$\Rightarrow F_0 = \frac{B_0}{B_T} \xrightarrow{} e^{-rT}\quad Q^B(S_T < K)\, K$$

recall that $\frac{dS_t}{S_t} = r\, dt + \sigma\, dW_t^B$

$$\Rightarrow S_T = S_0\, e^{(r - \frac{1}{2}\sigma^2)T + \sigma W_T^B}$$

$$\Rightarrow F_0 = e^{-rT} K\, \Phi\left(\frac{\log(K/S_0) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right)$$

$$G_0 = \mathbb{E}^{Q^S}\left[S_T\, \mathbb{1}_{S_T<K} \right]$$

$$\frac{G_0}{S_0} = \mathbb{E}^{\mathbb{Q}^S}\left[\frac{S_T \, \mathbb{1}_{S_T < K}}{S_T} \right]$$

$$= \mathbb{Q}^S (S_T < K)$$

$$dW_t^S = -\sigma \, dt + dW_t^B$$

$$\Rightarrow \quad W_T^B = \sigma T + W_T^S$$

$$\Rightarrow \quad S_T = S_0 \, e^{(r + \frac{1}{2}\sigma^2) T + \sigma W_T^S}$$

$$\Rightarrow \quad G_0 = S_0 \, \Phi\left(\frac{\log(K/S_0) - (r + \frac{1}{2}\sigma^2) T}{\sigma \sqrt{T}} \right)$$

suppose $r = (r_t)_{t \geq 0}$ satisfies the SDE

$$dr_t = \kappa(\theta - r_t)\,dt + \sigma\,dW_t^{B}$$

$$dB_t = r_t B_t\,dt$$

Ornstein-Uhlenbeck process
(Vasicek model)

bond-value $P(T) = \left(P_t(T)\right)_{t \geq 0}$ T-maturity

$$\frac{P_t(T)}{B_t} = \mathbb{E}_t^{\mathbb{Q}^B}\left[\frac{1}{B_T}\right]$$

$$\Rightarrow P_t(T) = \mathbb{E}_t^{\mathbb{Q}^B}\left[\frac{B_t}{B_T}\right] = \mathbb{E}_t^{\mathbb{Q}^B}\left[e^{-\int_t^T r_s\,ds}\right]$$

<u>NB:</u> $\mathcal{L}^r = \kappa(\theta - r)\,\partial_r + \frac{1}{2}\sigma^2\,\partial_{rr}$

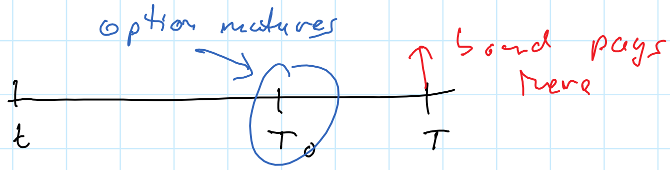
and so the model is affine.

and so, $P_t(T) = e^{A(t;T) - B(t;T)\,r_t}$

for some fn. A, B <u>deterministic.</u>

$\left(B \geq 0,\ B \xrightarrow[t \uparrow T]{} 0,\ A \xrightarrow[t \uparrow T]{} 0\right)$

How do we value a call option on a bond.



$$V_t = \mathbb{E}^{\mathbb{Q}^B}\left[(P_T(T) - K)_+\right]$$

$$\frac{V_t}{B_t} = \mathbb{E}_t^{\mathbb{Q}^B}\left[\frac{(P_{T_0}(T) - K)_+}{B_{T_0}}\right]$$

$$V_t = \mathbb{E}^{\mathbb{Q}^B}\left[e^{-\int_t^{T_0} r_s\,ds}\,(P_{T_0}(T) - K)_+\right]$$

$$P_{T_0}(T) = e^{A(T_0;T) - B(T_0;T)\,r_{T_0}}$$

this hard! So let's try something else..

$$\frac{V_t}{P_t(T_{\cancel{k}\,0})} = \mathbb{E}^{\mathbb{Q}^{T_{\cancel{k}\,0}}}\left[\frac{(P_{T_0}(T) - K)_+}{P_{T_0}(T_{\cancel{k}\,0})}\right]$$

$$P_{T_0}(T_0) \to 1$$

$$= \mathbb{E}^{\mathbb{Q}^{T_0}}\left[(P_{T_0}(T) - K)_+\right]$$

$$P_{T_0}(T) = \frac{P_{T_0}(T)}{P_{T_0}(T_0)}$$

$$X_t \triangleq \frac{P_t(T)}{P_t(T_0)} \xrightarrow[t \uparrow T_0]{} P_{T_0}(T)$$

X is a $\mathbb{Q}_{T_0}$ - mtg!

so $$\frac{dX_t}{X_t} = \sigma_t(T)\,dW_t^{T_0} - \sigma_t(T_0)\,dW_t^{T_0}$$

$$\Big[\qquad X_t = e^{\overbrace{(A(t,T) - A(t,T_0))}^{a(t)} + \overbrace{(B(t,T_0) - B(t,T))}^{b(t)}\,r_t}$$

$$dX_t = (\partial_t + \mathcal{L})\left(e^{a(t) + b(t) r_t}\right) dt$$
$$+ \, b(t)\, X_t\, \sigma\, dW_t^{B}$$
$$- \, b(t)\, X_t\, \sigma\, dW_t^{T_0} \Big]$$

$$\boxed{\frac{dX_t}{X_t} = \underbrace{\sigma\big(B(t,T_0) - B(t,T)\big)}_{u(t)}\, dW_t^{T_0}}$$

$$X_{T_0} = X_t \exp\Big\{ -\tfrac{1}{2}\int_t^{T_0} u^2(s)\, ds + \underbrace{\int_t^{T_0} u(s)\, dW_s^{T_0}}_{\sim \mathcal{N}(0;\, \Sigma^2)} \Big\}$$

$$\Sigma^2 = \mathbb{E}\Big[\Big(\int_t^{T_0} u(s)\, dW_s^{T_0}\Big)^2\Big]$$
$$= \mathbb{E}\Big[\int_t^{T_0} u^2(s)\, ds\Big]$$
$$= \int_t^{T_0} u^2(s)\, ds$$

$$\Rightarrow \quad V_t = P_t(T_0)\, \underbrace{\mathbb{E}^{\mathbb{Q}^{T_0}}\big[(X_{T_0} - K)_+\big]}_{}$$

like B-S with zero interest & vol^2 = Σ^2
i.e. $\sigma^2 T \to \Sigma^2$

$$\Rightarrow \quad F = \mathbb{E}_0^{\mathbb{Q}^{T_0}}\left[X_{T_0} \mathbb{1}_{X_{T_0} > K} \right]$$

$$G = K \, \mathbb{Q}_0^{T_0}(X_{T_0} > K)$$

$$F = \mathbb{E}_0^{\mathbb{Q}^{T_0}}\left[\frac{X_{T_0}}{\underbrace{\mathbb{E}_0^{\mathbb{Q}^{T_0}}[X_{T_0}]}_{X_0}} \mathbb{1}_{X_{T_0} > K} \right] \cdot \underbrace{\mathbb{E}_0^{\mathbb{Q}^{T_0}}[X_{T_0}]}_{X_0}$$

$$\frac{d\mathbb{Q}^*}{d\mathbb{Q}^{T_0}} = \frac{X_{T_0}}{X_0}$$

$$= X_0 \, \mathbb{Q}^*(X_{T_0} > K) .$$