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$$B_{n+1} = B_n e^{r \Delta t}$$

$$A_n \begin{cases} \xrightarrow{p} A_n e^{\sigma\sqrt{\Delta t}} \\ \xrightarrow{1-p} A_n e^{-\sigma\sqrt{\Delta t}} \\ \xrightarrow{\lambda \Delta t} 0 \end{cases}$$

$\lambda \Delta t$ — hazard rate

$$p = (1 - \lambda \Delta t)\, \frac{1}{2}\left(1 + \frac{\mu - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right)$$

$$1-p = (1 - \lambda \Delta t)\, \frac{1}{2}\left(1 - \frac{\mu - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right)$$

$\Delta t = T/n$

$0 \quad \cdots \quad T$

$$A_T = A_0 \exp\left\{ \sigma\sqrt{\Delta t} \sum_{m=1}^{n} x_m \right\}$$

x_m: ± 1 Bernoulli

$$\times \; \mathbb{1}_{\tau > T}$$

τ: 1st arrival of a Poisson process with intensity λ

$$\mathbb{1}_{\tau > T} = (1 - y_1)(1 - y_2) \cdots (1 - y_n)$$

$$\mathbb{P}(y_n = 0) = 1 - \lambda \Delta t$$

$$\mathbb{P}(y_n = 1) = \lambda \Delta t$$

$$X = \sigma\sqrt{\Delta t} \sum_{m=1}^{n} x_m \xrightarrow[n \to +\infty]{d} \mathcal{N}\left((\mu - \tfrac{1}{2}\sigma^2)T - \lambda T \,;\; \sigma^2 T \right)$$

Pricing measure is <u>not</u> unique

$$A_n \begin{cases} \xrightarrow{(1-\lambda^{\mathbb{Q}}\Delta t)\, q} A_n e^{\sigma\sqrt{\Delta t}} \\ \xrightarrow{(1-\lambda^{\mathbb{Q}}\Delta t)(1-q)} A_n e^{-\sigma\sqrt{\Delta t}} \end{cases}$$

$$A_n \begin{cases} \quad \\ A_n e^{-\sigma\sqrt{\Delta t}} & (1-\lambda^{\mathbb{Q}}\Delta t)(1-q) \\ 0 & \lambda^{\mathbb{Q}}\Delta t \end{cases}$$

$\lambda^{\mathbb{Q}} \longrightarrow \lambda$ for notational ease.

mtg condition: bank account as numeraire.

$$1 = e^{\sigma\sqrt{\Delta t} - r\Delta t} \cdot \underbrace{(1-\lambda\Delta t)}_{e^{-\lambda\Delta t}} q + e^{-\sigma\sqrt{\Delta t} - r\Delta t} (1-\lambda\Delta t)(1-q)$$

$$\frac{e^{r\Delta t}}{1-\lambda\Delta t}$$

$$q = \frac{e^{(r+\lambda)\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}$$

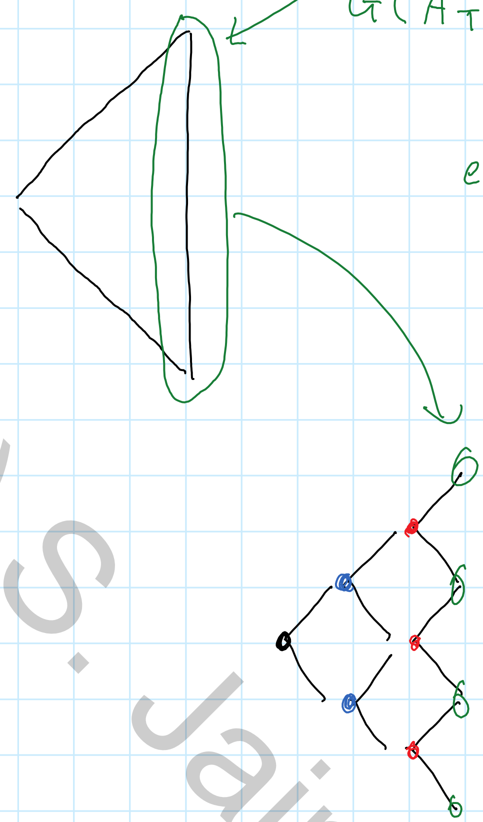
$$= \frac{1}{2}\left(1 + \frac{r+\lambda-\frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right) + \ldots$$

$$A_T \overset{d}{=} A_0 e^{X} \cdot \mathbb{1}_{\tau > T}$$

$$X \underset{n\to\infty}{\overset{\mathbb{Q}}{\sim}} \mathcal{N}\left((r+\lambda-\tfrac{1}{2}\sigma^2)T \,;\, \sigma^2 T \right)$$

$$\mathbb{E}^{\mathbb{Q}}[A_T / A_0 \mid \tau > T] = e^{(r+\lambda)T}$$

$$\mathbb{E}^{\mathbb{Q}}[A_T / A_0] = e^{rT}$$



$G(A_T)$ European options pay $G(A_T)$ @ T.

e.g. $G(x) = (x-K)_+$ call

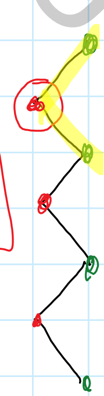
$(K-x)_+$ put

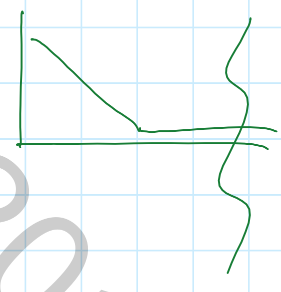
American option allows you to exercise your right at any point $t \in [0,T]$ and then receive $G(A_T)$.

$V^E_{m,k} = G(A_{m,k})$

$$\frac{V^H_{m,k}}{B_{m,k}} = \mathbb{E}^{\mathbb{Q}}\left[\frac{V_{m+1}}{B_{m+1}} \Big| \mathcal{F}_m\right]$$

$V_{m,k} = \max\left(V^H_{m,k}, V^E_{m,k}\right)$

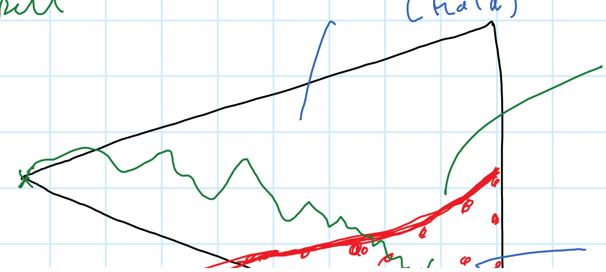




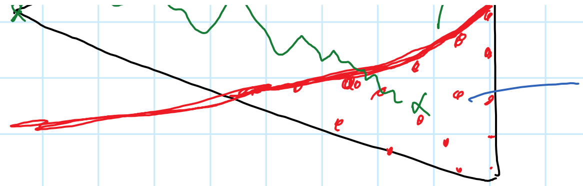
For a put

continuation region (hold)

optimal exercise boundary



stopping region



stopping region (exercise)

$$(K - A)_+ \hookrightarrow 0$$

* Underlying source of uncertainty $X = (X_t)_{t \geq 0}$, and X satisfies SDE

$$dX_t = \mu^X_t \, dt + \sigma^X_t \, dW_t$$

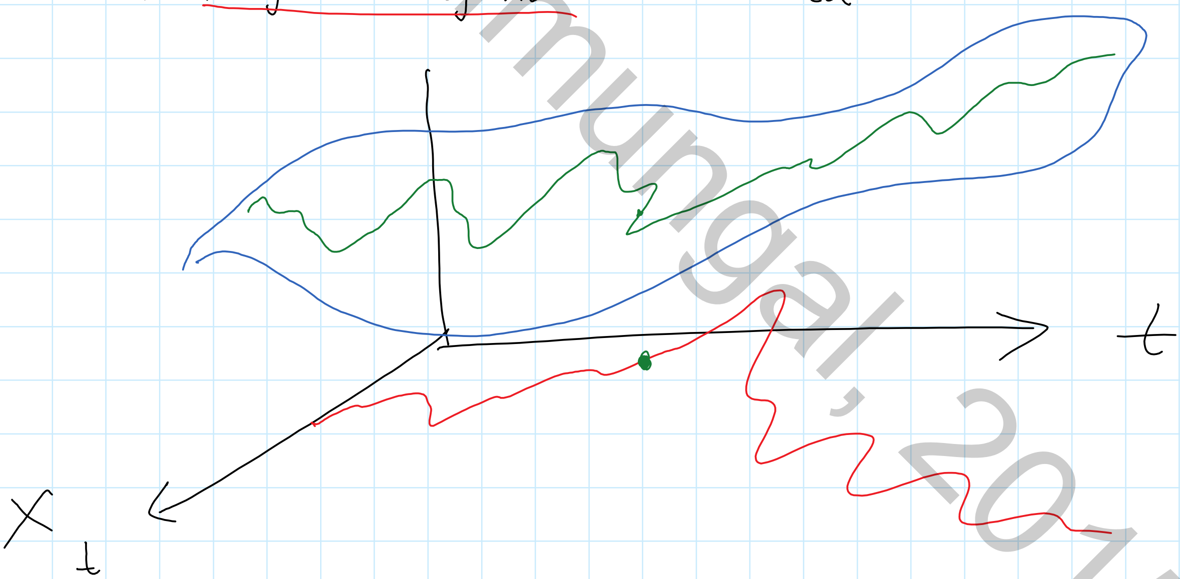
$W = (W_t)_{t \geq 0}$ a $\mathbb{P}$-Brownian motion

$\mu^X = (\mu^X_t)_{t \geq 0}$, $\sigma^X = (\sigma^X_t)_{t \geq 0}$

$\exists$ fns s.t. $\mu^X_t = \mu^X(t, X_t)$
$\sigma^X_t = \sigma^X(t, X_t)$

X may or may not be traded

$\sigma(t, x) = x^2 \sigma t t$



* Bank account $B = (B_t)_{t \geq 0}$ and

$$dB_t = r_t \, B_t \, dt$$

$r = (r_t)_{t \geq 0}$ is a short-rate process

$$r_t = r(t, X_t)$$

What is the value of options on X?

price is $F = (F_t)_{t\geq 0}$ and $F_T = F(X_T)$.

$$g = (g_t)_{t\geq 0} \quad \text{and} \quad g_T = G(X_T)$$

\# of B

$\ast$ self-financing strategy $(\alpha, \beta) = (\alpha_t, \beta_t)_{t\geq 0}$

\# of g $\quad -1$ in F.

$$f_t = F(t, X_t), \quad g_t = g(t, X_t)$$

$f : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}$

$f(t, x)$

Dynamic Hedging:

- $V_0 = 0$

self-financing

$$V_t = \alpha_t g_t + \beta_t B_t - f_t$$

$$dV_t = d\alpha_t \, g_t + \alpha_t \, dg_t + d[\alpha, g]_t$$

$$+ d\beta_t \, B_t + \beta_t \, dB_t + \underbrace{d[\beta, B]_t}_{0}$$

$$- df_t$$

self-financing condition

$$= \alpha_t \, dg_t + \beta_t \, dB_t - df_t$$

can write: $$dg_t = g_t \mu_t^g \, dt + g_t \sigma_t^g \, dW_t$$

$$df_t = f_t \mu_t^f \, dt + f_t \sigma_t^f \, dW_t$$

$$g_t = g(t, X_t) \qquad g_t$$

$$dg_t = \Big[\partial_t g(t,X_t) + \mu^X(t,X_t)\, \partial_x g(t,X_t) + \tfrac{1}{2}(\sigma^X(t,X_t))^2 \, \partial_{xx} g(t,X_t) \Big] dt$$

$$+ \underbrace{\sigma^X(t,X_t)\, \partial_x g(t,X_t)}_{\sigma_t^g \, g_t} \, dW_t$$

$\mu_t^g \, g_t$

$$dV_t = \alpha_t \left[\mu_t^g \, g_t \, dt + \sigma_t^g \, g_t \, dW_t \right]$$

$$+ \beta_t \, r_t \, B_t \, dt$$

$$- \left[\mu_t^f \, f_t \, dt + \sigma_t^f \, f_t \, dW_t \right]$$

$$= \left[\alpha_t \, \mu_t^g \, g_t + r_t \, \beta_t B_t - \mu_t^f \, f_t \right] dt$$

A_t

$$+ \left(\alpha_t \, \sigma_t^g \, g_t - \sigma_t^f \, f_t \right) dW_t$$

locally remove risk $= 0$

$$\boxed{\alpha_t = \frac{\sigma_t^f \, f_t}{\sigma_t^g \, g_t}}$$

since $V_0 = 0$ & $dV_t = A_t \, dt$

A_t is $\mathcal{F}_t$-measurable

to avoid arbitrage we must have $A_t = 0$

$\Rightarrow dV_t = 0 \Rightarrow V_t = 0 \Rightarrow \alpha_t g_t + \beta_t B_t = f_t$

$$\Rightarrow \beta_t B_t = f_t - \alpha_t g_t$$

$$= f_t \left(1 - \frac{\sigma_t^f}{\sigma_t^g} \right)$$

$$A_t = \frac{\sigma_t^f}{\sigma_t^g} \mu_t^g f_t + r_t f_t \left(1 - \frac{\sigma_t^f}{\sigma_t^g} \right) - \mu_t^f f_t = 0$$

$$\Rightarrow \boxed{\frac{\mu_t^f - r_t}{\sigma_t^f} = \frac{\mu_t^g - r_t}{\sigma_t^g}} = \lambda_t^X$$

Market price of risk.

$$\text{e.g. } \lambda_t^X = a(t) + b(t) X_t$$

if X is traded

$$\lambda_t^X = \frac{\mu_t^X/X_t - r_t}{\sigma_t^X/X_t}$$

$$\frac{\mu_t^f - r_t}{\sigma_t^f} = \lambda_t^X$$

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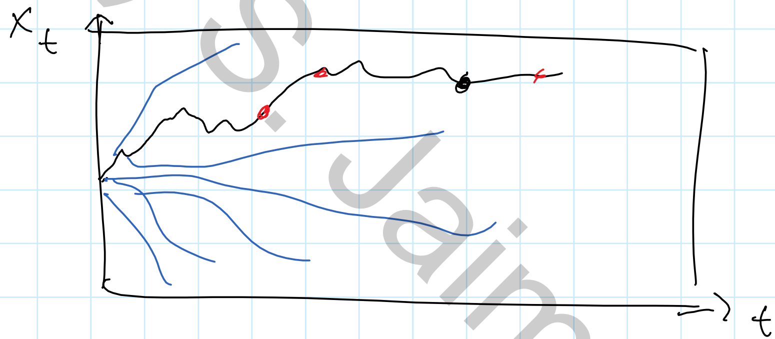
$$\Rightarrow \quad f_t \mu_t^f = f_t r_t + \sigma_t^f f_t \lambda_t^X$$

$$\Rightarrow \quad \partial_t f_t + \mu_t^X \partial_x f_t + \tfrac{1}{2}(\sigma_t^X)^2 \partial_{xx} f_t$$

$$= f_t r_t + \sigma_t^X \partial_x f_t \lambda_t^X$$

$$\Rightarrow \quad \left\{ \partial_t + (\mu_t^X - \sigma_t^X \partial_x) + \tfrac{1}{2}(\sigma_t^X)^2 \partial_{xx} \right\} f_t = r_t f_t$$



must hold $\forall$ paths and therefore

$$\boxed{\begin{aligned} &\left\{ \partial_t + \left(\mu^X(t,x) - \sigma^X(t,x)\,\lambda^X(t,x)\right) \partial_x \right. \\ &\quad \left. + \tfrac{1}{2}\left(\sigma^X(t,x)\right)^2 \partial_{xx} \right\} f(t,x) = r(t,x)\, f(t,x) \\ &\qquad f(T,x) = F(x) \end{aligned}}$$

Generalized Black-Scholes PDE.

$$\left(\partial_t + r\,x\,\partial_x + \tfrac{1}{2}\sigma^2 x^2 \partial_{xx}\right) f = r f$$

if X is traded, then we have that

$$\mu^X(t,x) - \sigma^X(t,x)\,\lambda^X(t,x) = r(t,x)\,x$$

$\Rightarrow$

$$\boxed{\begin{aligned}&\left(\partial_t + r(t,x)\,x\,\partial_x + \tfrac{1}{2}\left(\sigma^x(t,x)\right)^2 \partial_{xx}\right) f(t,x) = r(t,x)\, f(t,x)\\ &\qquad\qquad\qquad\qquad\qquad\qquad f(T,x) = F(T)\end{aligned}}$$

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$$[X, Y]_t = \lim_{\|\pi\| \downarrow 0} \sum_k \Delta X_k \, \Delta Y_k$$

covariation

e.g. X, Y are correlated (ρ) B. motion

$$[X, Y]_t = \rho \, t$$