$$G(X_T, Y_T) = X_T \, \mathbb{1}_{X_T > \alpha Y_T}$$

$$dX_t = r X_t \, dt + \sigma X_t \, dW_t^X$$

$$dY_t = r Y_t \, dt + \eta Y_t \, dW_t^Y$$

$$V_0 = \mathbb{E}^{\mathbb{Q}}\left[e^{-rT} X_T \, \mathbb{1}_{X_T > \alpha Y_T} \right]$$

$$\frac{V_0}{X_0} = \mathbb{E}^{\mathbb{Q}^X}\left[\frac{\cancel{X_T} \, \mathbb{1}_{X_T > \alpha Y_T}}{\cancel{X_T}} \right]$$

$$= \mathbb{Q}^X (X_T > \alpha Y_T)$$

$$= \mathbb{Q}^X \left(\frac{Y_T}{X_T} < \alpha^{-1} \right)$$

$$Z_t = \frac{Y_t}{X_t}$$ is a $\mathbb{Q}^X$-mtg!

$$dZ_t = (\cdots)\overset{\to 0}{} \, dt + (\text{vol of } Y_t) - (\text{vol of } X_t)$$

s.c

$$= (\eta \, d\hat{W}_t^Y - \sigma \, d\hat{W}_t^X) Z_t$$

$$Z_t = f(X_t, Y_t) \qquad f(x,y) = y/x$$

$$dZ_t = (\cdots) \, dt + \underbrace{\partial_x f(X_t, Y_t)}_{\to (-Y_t/X_t^2)} \cdot \sigma X_t \, dW_t^X$$

$$+ \underbrace{\partial_y f(X_t, Y_t)}_{\to (1/X_t)} \cdot \eta Y_t \, dW_t^Y$$

$$= Z_t \left(\eta \, d\hat{W}_t^Y - \sigma \, d\hat{W}_t^X \right)$$

$$= Z_t \big(\underbrace{\eta\, d\hat{W}^y_t - \sigma\, d\hat{W}^x_t}_{\mathbb{Q}^x\text{- B.motion}} \big)$$

$$V_0 = X_0\, \mathbb{Q}^x\left(Z_T < \alpha^{-1} \right)$$

$$\phi\, \hat{W}_T \overset{d}{=} \eta\, \hat{W}^y_T - \sigma\, \hat{W}^x_T \sim \mathcal{N}\left(0\,;(\sigma^2+\eta^2-2\sigma\eta\rho)T\right)$$

$$\Rightarrow \quad \phi^2 = \sigma^2 + \eta^2 - 2\sigma\eta\rho$$

and $Z_T = Z_0\, e^{-\frac{1}{2}\phi^2 T + \phi\, \hat{W}_T}$

$$dZ_t = Z_t\, \phi\, d\hat{W}_t$$

$$d(\log Z_t) = -\frac{1}{2}\phi^2\, dt + \phi\, d\hat{W}_t$$

$$V_0 = X_0\, \mathbb{Q}^x\left(Z_0\, e^{-\frac{1}{2}\phi^2 T + \phi\, \hat{W}_T} < \alpha^{-1} \right)$$

$$= X_0\, \mathbb{Q}^x\left(\underset{\sim \sqrt{T}\hat{Z}}{\hat{W}_T} < \frac{-\log(\alpha Z_0) + \frac{1}{2}\phi^2 T}{\phi} \right)$$

$$= X_0\, \Phi\left(\frac{-\log(\alpha Z_0) + \frac{1}{2}\phi^2 T}{\phi\sqrt{T}} \right)$$

$$dZ_t = Z_t(\eta\, d\hat{W}^y_t - \sigma\, d\hat{W}^x_t)$$

$f(z) = \log z$

$$df_t = \Big(0 \cdot \partial_z(\log Z_t) + \frac{1}{2}\overset{\phi^2}{(\eta^2 + \sigma^2 - 2\sigma\rho\eta)}\, Z_t^2 \cdot \partial_{zz}\log Z_t \Big)\, dt + \partial_z \log Z_t \cdot Z_t(\eta\, d\hat{W}^y_t - \sigma\, d\hat{W}^x_t)$$

$$= -\frac{1}{2}\phi^2\, dt + \eta\, d\hat{W}^y_t - \sigma\, d\hat{W}^x_t$$

$$Z_T = Z_0 \exp\{ -\tfrac{1}{2}\phi^2 T + \underbrace{\eta \hat{W}^Y_T - \sigma \hat{W}^X_T}_{\sim \mathcal{N}(0\,;\,\phi^2 T)} \}$$

m.r. rate, m.r. level, vol

$$dr_t = \kappa(\theta - r_t)\,dt + \sigma\,dW_t$$

← continuous analog of AR(1)

Vasicek Model

$$r_t = \theta_t + x_t$$

$$dx_t = -\kappa x_t\,dt + \sigma\,dW_t$$

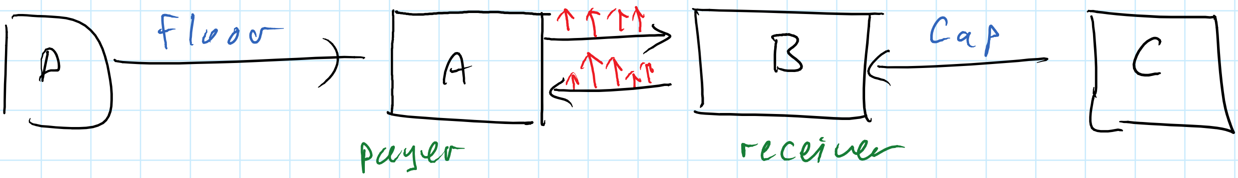
$$P_t(T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s\,ds} \right] = e^{A_t - B_t r_t}$$

affine solutions.

$$dr_t = \kappa_t(\theta_t - r_t)\,dt + \eta_t\sqrt{a_t + b_t r_t}\,dW_t$$

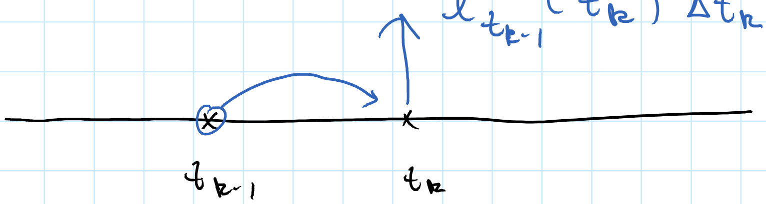
$$\mathcal{L}^{\wedge} = \kappa_t(\theta_t - r)\,\partial_r + \frac{1}{2}\eta_t^2(a_t + b_t r)\,\partial_{rr}$$

Interest Rate Cap



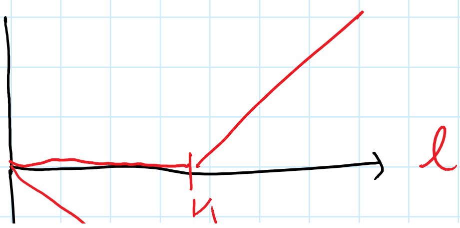
Caps made of Caplets

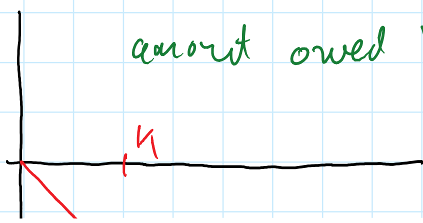
$\ell_{t_{k-1}}(t_k)\,\Delta t_k$



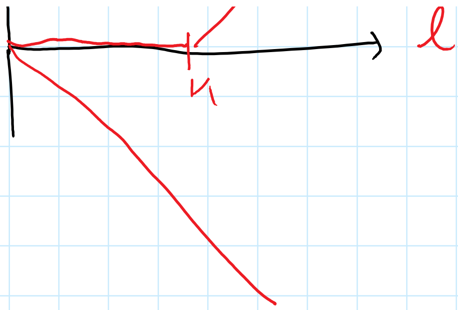
Caplet pays at t_k

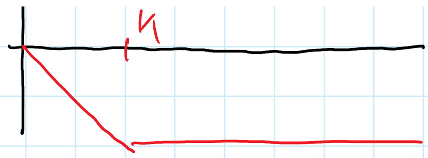
$$\left(\ell_{t_{k-1}}(t_k) - K\right)_+ \Delta t_k$$





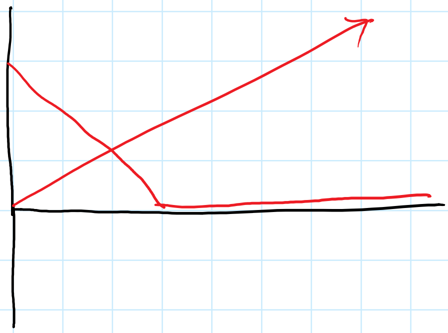
amount owed by B is "capped".

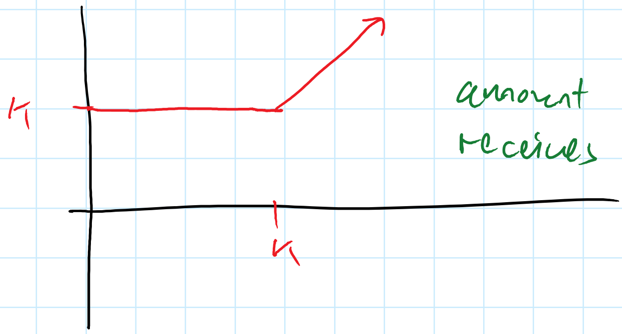




Floors are made of floorlets and each pays

$$(K - \ell_{t_{k-1}}(t_k))_+ \Delta t_k$$





amount A receives is floored

Value claim paying $\left(l_{t_{k-1}}(t_k) - K\right)_+ \Delta t_k$ @ t_k.

define

$$l_t^{(k)} \triangleq \frac{1}{\Delta t_k}\left(\frac{P_t(t_{k-1})}{P_t(t_k)} - 1\right)$$

$$\xrightarrow[t \to t_{k-1}]{a.s.} l_{t_{k-1}}(t_k) = \frac{1}{\Delta t_k}\left(\frac{1}{P_{t_{k-1}}(t_k)} - 1\right)$$

$l_t^{(k)}$ is in fact the forward rate at t for the period $[t_{k-1}, t_k]$

$l_t^{(k)}$ is a $\mathbb{Q}^{t_k}$ - martingale

↳ using $P_t(t_k)$ as numeraire.

also note: $P_t(t_k) \xrightarrow[t \to t_k]{} 1$

hence: $$\frac{V_t}{P_t(t_k)} = \mathbb{E}^{\mathbb{Q}^{t_k}}\left[\frac{\Delta t_k\left(l^{(k)}_{t_{k-1}} - K\right)_+}{\underbrace{P_{t_k}(t_k)}_{1}}\right]$$

$$\Rightarrow \boxed{V_t = \Delta t_k P_t(t_k)\, \mathbb{E}^{\mathbb{Q}^{t_k}}\left[\left(l^{(k)}_{t_{k-1}} - K\right)_+\right]}$$

since $l_t^{(k)}$ is a $\mathbb{Q}^{t_k}$-mtg why not model as...

$$d l_t^{(k)} = \sigma^{(k)} \, l_t^{(k)} \, \underbrace{dW_t^{(k)}}_{\mathbb{Q}^{t_k}\text{- B.mtn.}} \quad (*)$$

Compare with Vasicek...

$$P_t(T) = e^{A_t^T - B_t^T r_t}$$

$$d P_t(T) = \underbrace{(\ldots)}_{r_t P_t(T)} dt - B_t^T \sigma . P_t(T) \, dW_t$$

$$\Rightarrow \quad \frac{dP_t(T)}{P_t(T)} = r_t \, dt - \sigma B_t^T \, dW_t$$

$$X_t^{(k)} \triangleq \frac{P_t(t_{k-1})}{P_t(t_k)}$$

$$d X_t^{(k)} = (\ldots) dt + X_t^{(k)} \Big(-\sigma B_t^{(k-1)} dW_t - (-\sigma B_t^{(k)} dW_t) \Big)$$

$$\frac{d X_t^{(k)}}{X_t^{(k)}} = \underbrace{\sigma (B_t^{(k)} - B_t^{(k-1)})}_{\sigma_t^{(k)}} \, dW_t^{(k)}$$

$\hookrightarrow$ $\mathbb{Q}^{t_k}$- B.mtn

recall that $\ell_t^{(k)} = \frac{1}{\Delta t_k} \left(X_t^{(k)} - 1 \right)$

$$\Rightarrow \quad d \ell_t^{(k)} = \frac{1}{\Delta t_k} d X_t^{(k)}$$

$$= \frac{1}{\Delta t_k} X_t^{(k)} \sigma_t^{(k)} dW_t^{(k)}$$

$$= \left(\frac{1}{\Delta t_k} + \ell_t^{(k)} \right) \sigma_t^{(k)} dW_t^{(k)}$$

$$\Rightarrow \quad \frac{d \ell_t^{(k)}}{\left(\ell_t^{(k)} + \frac{1}{\Delta t_k} \right)} = \sigma_t^{(k)} dW_t^{(k)}$$

inconsistent with (*)

$$V^{Cap} = \sum_k V_{caplet}^{(k)}$$

Black Model assumes $\sigma^{(k)} = \sigma$ & we have:

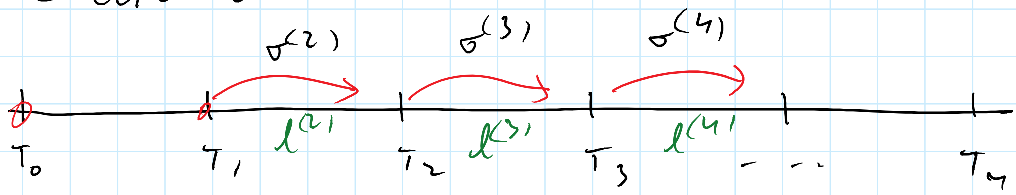
$$\psi^{(k)} = \mathbb{E}^{\mathbb{Q}^{t_k}}\left[\left(\ell^{(k)}_{t_{k-1}} - K \right)_+ \right] \Delta t_k$$

$$= \Delta t_k \left[\ell_0^{(k)} \, \Phi(d_+) - K \, \Phi(d_-) \right]$$

$$d_\pm = \frac{\log(\ell_0^{(k)}/K) \pm \frac{1}{2}\sigma^2 t_{k-1}}{\sigma \sqrt{t_{k-1}}}$$

$$V^{(k)}_{caplet} = P_t(t_k) \cdot \psi^{(k)}$$

Calibration



$$\begin{array}{ccc} \cancel{V^{(1)}}, & V^{(2)}, \; \dots, & V^{(n)} \\ \uparrow & \uparrow & \uparrow \\ \cancel{\sigma_B^{(1)}} & \sigma_B^{(2)} & \sigma_B^{(n)} \end{array}$$

$$\frac{d\ell_t^{(k)}}{\ell_t^{(k)}} = \sigma_t^{(k)} \, dW_t^{(k)}$$

$$V^{(2)} \Longrightarrow \sigma^{(2)} = \text{const on } (T_0, T_1)$$

$$V^{(3)} \Longrightarrow \sigma^{(3)} = \begin{cases} \text{const} & \text{on } (T_0, T_1) \\ \sigma^{(2)} & \text{on } (T_1, T_2) \end{cases}$$

$$V^{(4)} \Longrightarrow \sigma^{(4)} = \begin{cases} \text{const.} & \text{on } (T_0, T_1) \\ \sigma^{(3)} & \text{on } (T_1, T_2) \\ \sigma^{(2)} & \text{on } (T_2, T_3) \end{cases}$$

	period 1	2	3	4
$\sigma_t^{(2)} =$	$\sigma^{(2)}$	—	—	—

$$\sigma_t^{(3)} = \sigma^{(3)} \quad \sigma^{(2)} \quad - \quad -$$

$$\sigma_t^{(4)} = \sigma^{(4)} \quad \sigma^{(3)} \quad \sigma^{(2)} \quad -$$

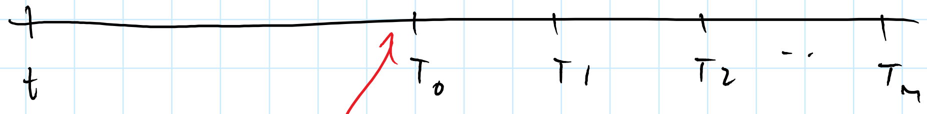
i.e. $\sigma_t^{(n)}$ is a fn only of time remaining.

only unknowns.

26 November 2014 15:58

Swap options: swaptions (payer or receiver)

strike = F.



1st swap reset & maturity of option.

swap-rate $\quad S_t = \dfrac{V_t^{fl}}{V_t^{fix}} = \dfrac{P_t(T_0) - P_t(T_n)}{\sum_{k=1}^{n} \Delta T_k \, P_t(T_k)}$

Clear that if $S_{T_0} > F$ exercise otherwise walk away.

if exercised then cashflow $= V_{T_0}^{fl} - V_{T_0}^{fix} \cdot F$

$$= V_{T_0}^{fix}\left(\frac{V_{T_0}^{fl}}{V_{T_0}^{fix}} - F\right)$$

$\therefore$ payoff at T_0 $= \left(V_{T_0}^{fl} - F\, V_{T_0}^{fix}\right)_+$

$$= \left(P_{T_0}(T_0) - P_{T_0}(T_n) - F\sum_{k=1}^{n} \Delta t_k \, P_{T_0}(T_k)\right)_+$$

$$= \left(1 - \sum_{k=1}^{n} c_k \, P_{T_0}(T_k)\right)_+$$

$$c_k = \begin{cases} F\Delta t_k, & k \neq n \\ 1 + F\Delta t_n, & k = n \end{cases}$$

For an affine model $P_t(T) = e^{A_t^T - B_t^T r_t}$, $(B_t^T \geq 0)$

$$r \nearrow \;\Rightarrow\; P \searrow$$

so find r^* s.t. $\quad 1 = \sum_{k=1}^{n} c_k \, P_{T_0}(r^*; T_k)$

$$\Rightarrow \quad G = \left(1 - \sum_{k=1}^{n} c_k P_{T_0}(r_{T_0}; T_k)\right) \mathbb{1}_{r_{T_0} > r^*}$$

$$= \left(\sum_{k=1}^{n} c_k \left(P_{T_0}(r^*; T_k) - P_{T_0}(r_{T_0}; T_k) \right) \right) \mathbb{1}_{r_{T_0} > r^*}$$

$$= \sum_{k=1}^{n} \left\{ c_k \left(P_{T_0}(r^*; T_k) - P_{T_0}(r_{T_0}; T_k) \right) \mathbb{1}_{r_{T_0} > r^*} \right\}$$

$$\mathbb{1}_{r_{T_0} > r^*} = \mathbb{1}_{P_{T_0}(r^*; T_k) > P_{T_0}(r_{T_0}; T_k)}$$

$$(A - B) \cdot \mathbb{1}_{A > B} = (A - B)_+$$

$$G = \sum_{k=1} c_k \left(\underbrace{P_{T_0}(r^*; T_k)}_{p^*(k)} - P_{T_0}(r_{T_0}; T_k) \right)_+$$

now value each bond option.