# Futures Contracts

06 November 2013 14:07

a claim H on a Futures contract which pays $\varphi(F_{T_0}(T))$ at T satisfies the PDE:

$$\begin{cases} \partial_t h + \frac{1}{2}\left(\sigma^F(t,F)\right)^2 \partial_{FF} h = r h \\ h(T_0, F) = \varphi(F) \end{cases}$$

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$$dF_t = \mu^F(t, F_t)\, dt + \sigma^F(t, F_t)\, dW_t$$

$\rightarrow$ $\mathbb{P}$ - B. mtn.

$(F_t(T))$

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Feynman-Kac then says that:

$$h(t, F) = \mathbb{E}^{\mathbb{Q}}\left[ \varphi(F_{T_0}) e^{-r(T_0 - t)} \mid F_t = F \right] \quad \left(\mathbb{E}_{t,F}[\cdot]\right)$$

where $F_t$ satisfies the SDE:

$$dF_t = \sigma^F(t, F_t)\, d\hat{W}_t$$

$\leftarrow$ $\mathbb{Q}$ - B. mtn.

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recall for a traded asset:

$$dX_t = r X_t\, dt + \sigma^X(t, X_t)\, d\hat{W}_t$$

$$\left(\text{if } dX_t = \mu^X_t\, dt + \sigma^X_t\, dW_t\right)$$

if non-traded

$$dX_t = (\mu^X_t - \lambda_t \sigma^X_t)\, dt + \sigma^X_t\, dW_t$$

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Note: $F_t - F_s = \int_s^t \sigma^F(u, F_u)\, d\hat{W}_u \qquad (s < t)$

$$\mathbb{E}^{\mathbb{Q}}[F_t - F_s \mid \mathcal{F}_s] = 0$$

$$\mathbb{E}[F_t - F_s \mid \mathcal{F}_t] = F_t - F_s$$

$\therefore$ F is a $\mathbb{Q}$-mtg!

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$$dF_t = 1\cdot dt + 2\cdot d\underset{\text{P-B.motion}}{W_t}$$

$$\varphi(F_1) = \mathbb{1}_{F_1>K} \qquad \& \qquad r=0$$

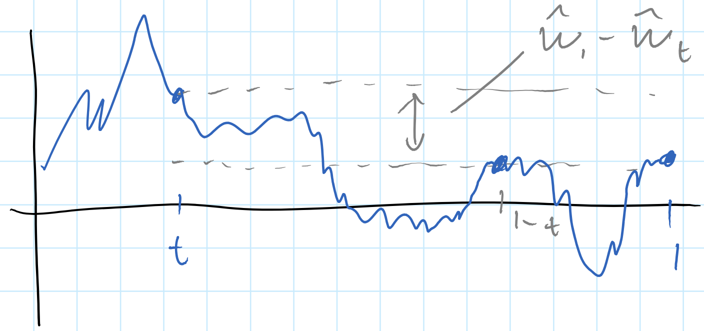
value the claim.

$$h(t,F) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(1-t)}\,\mathbb{1}_{F_1>K} \,\middle|\, F_t = F\right]$$

$$dF_t = 2\, d\underset{\text{Q-B.motion}}{\hat W_t}$$

$$\Rightarrow F_1 = F_t + 2\underbrace{(\hat W_1 - \hat W_t)}_{\hat W_{1-t}}$$

$$\hat W_{1-t} \overset{d}{=} \hat W_1 - \hat W_t$$




$$\mathbb{1}_{F_1>K} = \mathbb{1}_{F_t + 2(\hat W_1 - \hat W_t) > K}$$

$$= \mathbb{1}_{(\hat W_1 - \hat W_t) > \frac{K-F_t}{2}}$$

$$\Rightarrow h(t,F) = \mathbb{E}^{\mathbb{Q}}_{t,F}\left[\mathbb{1}_{F_1>K}\right]$$

$$= \mathbb{Q}_{t,F}\Big(\underbrace{\hat W_1 - \hat W_t}_{\sim \mathcal{N}(0,1-t)} > \frac{K-F}{2}\Big)$$

$$= \mathbb{Q}_{t,F}\left(Z > \frac{K-F}{2(1-t)^{1/2}}\right)$$

$$= \Phi\left(\frac{F-K}{2(1-t)^{1/2}}\right)$$

try : $dF_t(T) = \kappa(\theta - F_t(T))\,dt + \sigma e^{-\eta(T-t)}\,dW_t$

$$\varphi(F_{T_0}(T)) = \mathbb{1}_{F_{T_0}(T) > K}, \quad r = 0.$$

$$h(t,F) = \mathbb{E}^{\mathbb{Q}}\left[\varphi(F_{T_0})\, e^{-r(T_0 - t)} \,\middle|\, F_t = F\right]$$

where $F_t$ satisfies the SDE:

$$dF_t = \sigma^F(t, F_t)\, d\hat{W}_t \leftarrow \mathbb{Q}\text{- B. motion.}$$

$$dF_t = \sigma e^{-\eta(T-t)}\, d\hat{W}_t$$

$$\Rightarrow \quad F_{T_0} - F_t = \int_t^{T_0} \sigma e^{-\eta(T-u)}\, d\hat{W}_u$$

clearly, $(F_{T_0} - F_t)\big|_{\mathcal{F}_t}$ is normally distributed

$$\underset{\mathbb{Q}}{\sim} \mathcal{N}(0, \Sigma^2)$$

$$\Sigma^2 = \mathbb{E}^{\mathbb{Q}}_{t,F}\left[\left(\int_t^{T_0} \sigma e^{-\eta(T-u)}\, d\hat{W}_u\right)^2\right]$$

Ito issometry $\overset{\urcorner}{=} \mathbb{E}_{t,F}\left[\int_t^{T_0} \sigma^2 e^{-2\eta(T-u)}\, du\right]$

$$= \sigma^2 \int_t^{T_0} e^{-2\eta(T-u)}\, du$$

$$= \sigma^2 \, \frac{e^{-2\eta(T-T_0)} - e^{-2\eta(T-t)}}{2\eta}$$

$$h(t,F) = \mathbb{E}^{\mathbb{Q}}_{t,F}\left[\mathbb{1}_{F_{T_0}(T) > K}\right]$$

$$= \mathbb{Q}_{t,F}(F_{T_0} > K)$$

$$= \mathbb{Q}_{t,F}(F_{T_0} - F_t > K - F_t)$$

$\Sigma Z, \quad Z \underset{\mathbb{Q}}{\sim} N(0,1)$

$$= \mathbb{Q}_{t,F}\left( Z > \frac{K-F}{\Sigma} \right)$$

$$= \Phi\left( \frac{F-K}{\Sigma} \right)$$

# Measure Change

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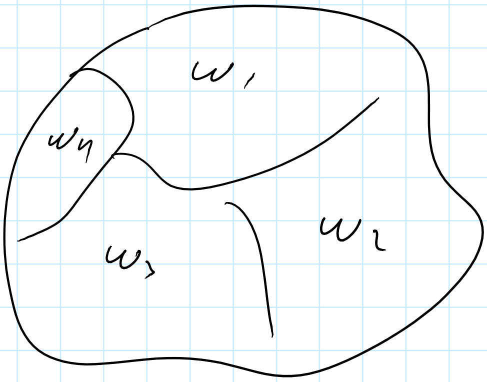
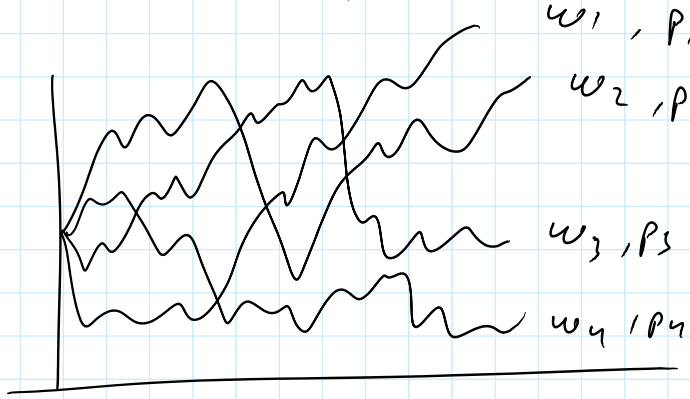



$(\Omega, \mathcal{F}, \mathbb{P})$

$\mathbb{P}(\omega_k) = p_k$



| | $\mathbb{P}$ | $\mathbb{P}^*$ |
|---|---|---|
| $\omega_1$ | $p_1$ | $p_1^*$ |
| $\omega_2$ | $p_2$ | $p_2^*$ |
| $\omega_3$ | $p_3$ | $p_3^*$ |
| $\omega_4$ | $p_4$ | $p_4^*$ |

$\mathbb{P}$ , $\mathbb{P}^*$

$\mathbb{P} \sim \mathbb{P}^*$ (equivalent)

$\mathbb{P}(\omega) \neq 0 \iff \mathbb{P}^*(\omega) \neq 0$

$$\mathbb{E}^{\mathbb{P}}[R] = \sum_k R(\omega_k)\, \mathbb{P}(\omega_k)$$

$$\mathbb{E}^{\mathbb{P}^*}[R] = \sum_k R(\omega_k)\, \mathbb{P}^*(\omega_k)$$

$$= \sum_{k:\, p_k^* > 0} R(\omega_k)\, \mathbb{P}^*(\omega_k)$$

$$= \sum_{k:\, p_k^* > 0} \left( R(\omega_k) \cdot \frac{\mathbb{P}^*(\omega_k)}{\mathbb{P}(\omega_k)} \right) \cdot \mathbb{P}(\omega_k)$$

$\hookrightarrow \mathcal{L}(\omega_k)$

$$= \sum_{k:\, p_k > 0} \mathcal{L}(\omega_k)\, \mathbb{P}(\omega_k)$$

$$= \mathbb{E}^{\mathbb{P}}[\mathcal{L}]$$

$$\boxed{\mathbb{E}^{\mathbb{P}^*}[R] = \mathbb{E}^{\mathbb{P}}\left[ R \cdot \frac{d\mathbb{P}^*}{d\mathbb{P}} \right]}$$

$$\boxed{\mathbb{E}^{\mathbb{P}^*}[R] = \mathbb{E}^{\mathbb{P}}\left[R \cdot \frac{d\mathbb{P}^*}{d\mathbb{P}}\right]}$$

So far: no arb $\iff \exists\ \mathbb{Q} \sim \mathbb{P}$ s.t.

traded assets $F$. satisfies.

$$\frac{F_t}{M_t} = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{F_s}{M_s}\right], \qquad (t<s)$$

if $$\frac{A_t}{M_t} = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{A_s}{M_s}\right] \quad \& \quad A_t > 0 \text{ a.s.}$$

then $\exists\ \mathbb{Q}^A$ s.t.

$$\frac{F_t}{A_t} = \mathbb{E}_t^{\mathbb{Q}^A}\left[\frac{F_s}{A_s}\right], \qquad (t<s)$$

( A is called a numeraire asset )

moreover, $$\frac{d\mathbb{Q}^A}{d\mathbb{Q}} = \frac{A_T/A_0}{M_T/M_0}$$ ( T is end of world )

Let's suppose that A satisfies the SDE:

$$\frac{dA_t}{A_t} = \mu_t^A\, dt + \sigma_t^A\, dW_t$$ $\mathbb{Q}$ - B. motion.

if A is a numeraire asset, then $\mu_t^A = r_t$!

$$d(A_t/M_t) = dA_t \frac{1}{M_t} + A_t\, d\left(\tfrac{1}{M_t}\right) + \underbrace{d\left[A, \tfrac{1}{M}\right]_t}_{0}$$

note: $M_t = e^{\int_0^t r_s ds}$, $\frac{1}{M_t} = e^{-\int_0^t r_s ds}$, $d\left(\frac{1}{M_t}\right) = -r_t M_t^{-1} dt$

$$\Rightarrow d(A_t/M_t) = (A_t/M_t)\left(\mu_t^A\, dt + \sigma_t^A\, dW_t\right)$$
$$+ (A_t/M_t)(-r_t\, dt)$$
$$= (A_t/M_t)\left[\underbrace{(\mu_t^A - r_t)}_{0}\, dt + \sigma_t^A\, dW_t\right]$$

$\hookrightarrow 0$ since $A_t/M_t$ is a $\mathbb{Q}$-mtg.

$$\eta_t \triangleq \mathbb{E}_t^{\mathbb{Q}}\left[\frac{d\mathbb{Q}^A}{d\mathbb{Q}}\right] = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{A_T/A_0}{\ }\right] = \frac{A_t/A_0}{\ }$$

$$\eta_t \triangleq \mathbb{E}_t^{\mathbb{Q}}\left[\frac{d\mathbb{Q}^A}{d\mathbb{Q}}\right] = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{A_T/A_0}{M_T/M_0}\right] = \frac{A_t/A_0}{M_t/M_0}$$

is a Doob-mtg $\mathbb{Q}$

$$\boxed{\frac{d\eta_t}{\eta_t} = \sigma_t^A \, dW_t}$$

solution of this is a Doléans-Dade exponential (stochastic exponential)

$$d(\ln \eta_t) = \frac{d\eta_t}{\eta_t} + \frac{1}{2}\left(-\frac{1}{\eta_t^2}\right)\cdot(\sigma_t^A \eta_t)^2 \, dt$$

$$= -\frac{1}{2}(\sigma_t^A)^2 dt + \sigma_t^A \, dW_t$$

$$\Rightarrow \ln \eta_t - \ln \underbrace{\eta_0}_{=1} = -\frac{1}{2}\int_0^t (\sigma_u^A)^2 \, du + \int_0^t \sigma_u^A \, dW_u$$

$$\Rightarrow \boxed{\eta_t = \exp\left\{-\frac{1}{2}\int_0^t (\sigma_u^A)^2 \, du + \int_0^t \sigma_u^A \, dW_u\right\}}$$

Girsanov's Thm:

$$W_t^A = -\int_0^t \sigma_u^A \, du + W_t$$

is a $\mathbb{Q}^A$ - B. mtn.

$$\left( dW_t^A = -\sigma_t^A \, dt + dW_t \right)$$

$\mathbb{Q}^A$: $\sim N(\sigma, 1)$

$W_1 = W_1^A + \sigma \, 1$

$\hookrightarrow \underset{\mathbb{Q}^A}{\sim} N(0,1)$

$$\frac{dA_t}{A_t} = r_t \, dt + \sigma_t^A \, dW_t$$

$$\frac{dB_t}{B_t} = r_t \, dt + \sigma_t^B \, dW_t$$

$$\begin{tikzcd} & \mathbb{Q}^A \arrow[d] \\ \mathbb{Q} \arrow[ur] \arrow[r] & \mathbb{Q}^B \end{tikzcd}$$

$$dW_t^A = -\sigma_t^A \, dt + dW_t$$

$$dW_t^A = -\sigma_t^A \, dt + dW_t$$

$$dW_t^B = -\sigma_t^B \, dt + dW_t$$

$$\Rightarrow \boxed{dW_t^A = -(\sigma_t^A - \sigma_t^B)\, dt + dW_t^B}$$

# Measure Change

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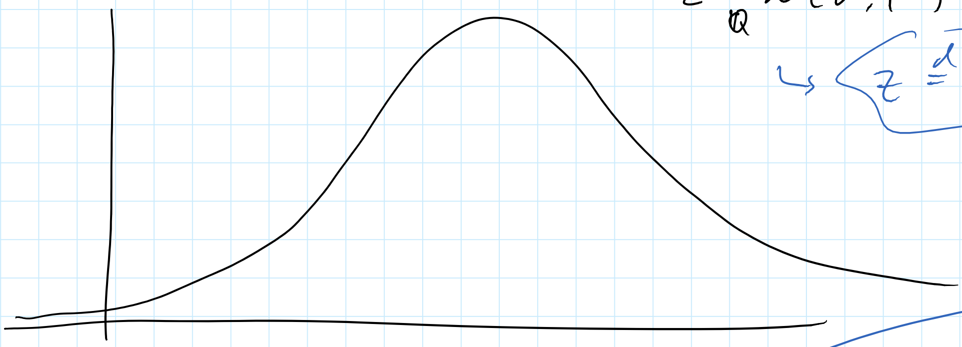
$Z \underset{\mathbb{Q}}{\sim} \mathcal{N}(0,1)$

$\hookrightarrow Z \overset{d}{=} W_1$



$\sigma \int_0^1 dW_u$

$$\frac{d\mathbb{Q}^*}{d\mathbb{Q}} = \exp\left\{-\frac{1}{2}\sigma^2 + \sigma Z\right\} \quad \text{clearly} > 0 \text{ a.s.}$$

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{d\mathbb{Q}^*}{d\mathbb{Q}}\right] = e^{-\frac{1}{2}\sigma^2} \cdot \mathbb{E}^{\mathbb{Q}}\left[e^{\sigma Z}\right] = 1$$

$\hookrightarrow e^{\frac{1}{2}\sigma^2}$

$$\mathbb{E}^{\mathbb{Q}^*}\left[e^{uZ}\right] = \mathbb{E}^{\mathbb{Q}}\left[e^{uZ} \cdot \frac{d\mathbb{Q}^*}{d\mathbb{Q}}\right]$$

$$= \mathbb{E}^{\mathbb{Q}}\left[e^{uZ - \frac{1}{2}\sigma^2 + \sigma Z}\right]$$

$$= e^{-\frac{1}{2}\sigma^2} \mathbb{E}^{\mathbb{Q}}\left[e^{(u+\sigma)Z}\right]$$

$$= e^{-\frac{1}{2}\sigma^2 + \frac{1}{2}(u+\sigma)^2}$$

$$= e^{u\sigma + \frac{1}{2}u^2}$$

so $\quad Z \underset{\mathbb{Q}^*}{\sim} \mathcal{N}(\sigma, 1)$

# Exchange Option

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suppose there are two traded assets A & B:

$$\frac{dA_t}{A_t} = \mu^A \, dt + \sigma^A \, dW_t$$

$$\frac{dB_t}{B_t} = \mu^B \, dt + \sigma^B \, dW_t$$

$\mathbb{P}$ - Brownian.

value an exchange option.

$$\varphi = \max(A_T - B_T, 0)$$

First:

$$\frac{dA_t}{A_t} = r \, dt + \sigma^A \, dW_t^{\mathbb{Q}}$$

$$\frac{dB_t}{B_t} = r \, dt + \sigma^B \, dW_t^{\mathbb{Q}}$$

$$f(t, A, B) = \mathbb{E}^{\mathbb{Q}}_{t,A,B}\left[ (A_T - B_T)_+ \, e^{-r(T-t)} \right]$$

$$\frac{f_t}{A_t} = \mathbb{E}^{\mathbb{Q}^A}_{t,A,B}\left[ \frac{(A_T - B_T)_+}{A_T} \right]$$

$$= \mathbb{E}^{\mathbb{Q}^A}_{t,A,B}\left[ \left(1 - \frac{B_T}{A_T}\right)_+ \right]$$

$X_t = \dfrac{B_t}{A_t}$ is a $\mathbb{Q}^A$ - mtg!

$$\frac{dX_t}{X_t} = (\sigma^B - \sigma^A) \, dW_t^A$$

$$d\left(\frac{1}{A_t}\right) = -\frac{dA_t}{A_t^2} + \frac{1}{2}\frac{2}{A_t^3}(\sigma^A A_t)^2 \, dt$$

$$= \left(\frac{1}{A_t}\right)\left(-\left(r\,dt + \sigma^A\,dW_t^{Q}\right)\right) + \left(\frac{1}{A_t}\right)(\sigma^A)^2\,dt$$

$$= \left(\frac{1}{A_t}\right)\left\{ \left((\sigma^A)^2 - r\right) dt - \sigma^A\,dW_t^{Q} \right\}$$

$$d\left(B_t \frac{1}{A_t}\right) = \frac{dB_t}{A_t} + B_t\, d\left(\frac{1}{A_t}\right) + d[B, \tfrac{1}{A}]_t$$

$\hookrightarrow \frac{B_t}{A_t} d[\sigma^B W, -\sigma^A W]_t$

$$= \frac{B_t}{A_t}\Big\{ \left(r\,dt + \sigma^B\,dW_t^{Q}\right)$$

$$+ \left((\sigma^A)^2 - r\right) dt - \sigma^A\,dW_t^{Q}$$

$$- \sigma_A \sigma_B\,dt \Big\}$$

$$= \frac{B_t}{A_t}\left\{ \left((\sigma^A)^2 - \sigma_A\sigma_B\right) dt + (\sigma^B - \sigma^A)\,dW_t^{Q} \right\}$$

$dW_t^{Q} = dW_t^A + \sigma^A\,dt$

$$\Rightarrow \quad \frac{dX_t}{X_t} = C\,dt + (\sigma^B - \sigma^A)\,dW_t^A$$

$$C = \left((\sigma^A)^2 - \sigma_A\sigma_B\right) + \sigma^A(\sigma^B - \sigma^A)$$

$$= 0$$

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$$\text{so } f_t = A_t\left[ \Phi(-d_-) - \frac{B_t}{A_t} \cdot \Phi(-d_+) \right]$$

$$d_\pm = \frac{\ln(B_t/A_t) \pm \frac{1}{2}(\sigma^B - \sigma^A)^2 (T-t)}{|\sigma^B - \sigma^A|\sqrt{T-t}}$$