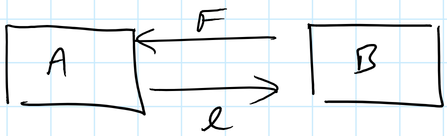
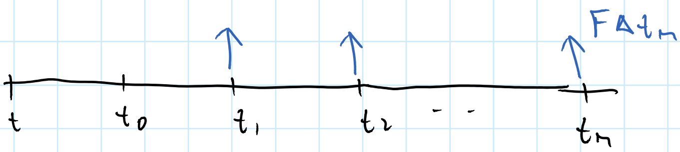
Caps / Swaption
swap options

Cap:

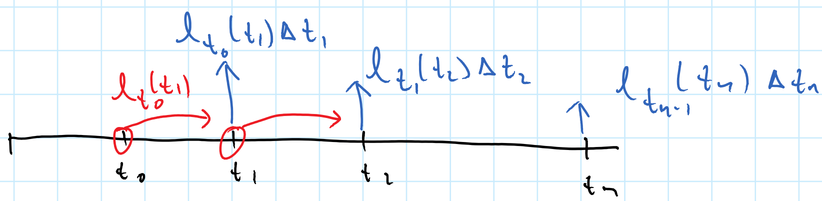
cap made of caplets



IRS

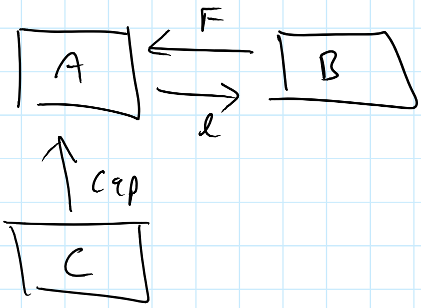


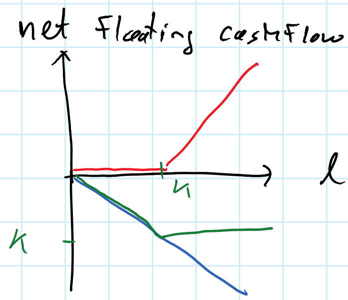
Fixed

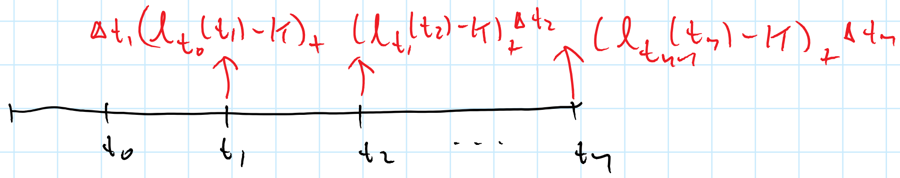


Floating

$$P_{t_{k-1}}(t_k) = \left(1 + \Delta t_k \cdot l_{t_{k-1}}(t_k)\right)^{-1} \Rightarrow l_{t_{k-1}}(t_k) = \frac{1}{\Delta t_k}\left[\frac{1}{P_{t_{k-1}}(t_k)} - 1\right]$$







each one of these are called caplets

Let $g_t^{(k)}$ denote the price of a caplet maturing at t_k

Let $g_t^{(k)}$ denote the price of a caplet maturing at t_k and set at t_{k-1}.

$$\frac{g_t^{(k)}}{M_t} = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{\Delta t_k \left(L_{t_{k-1}}(t_k) - K\right)_+}{M_{t_k}} \right]$$

instead use the t_k - bond as numeraire:

$$\frac{g_t^{(k)}}{P_t(t_k)} = \mathbb{E}_t^{\mathbb{Q}_k}\left[\frac{\left(L_{t_{k-1}}(t_k) - K\right)_+}{\underbrace{P_{t_k}(t_k)}_{\to 1}} \right] \Delta t_k$$

recall $L_{t_{k-1}}(t_k) = \frac{1}{\Delta t_k}\left[\frac{1}{P_{t_{k-1}}(t_k)} - 1 \right]$

$$X_t = \frac{P_t(t_{k-1})}{P_t(t_k)} \xrightarrow[t \uparrow t_{k-1}]{} \frac{1}{P_{t_{k-1}}(t_k)}$$

Moreover, X is a $\mathbb{Q}_k$ - mtg.

In the Vasicek model we had: $P_t(T) = e^{A_t(T) - B_t(T) r_t}$

$$\frac{dX_t}{X_t} = (\text{vol } t_{k-1}) - (\text{vol } t_k)$$

and we know that $\frac{dP_t(T)}{P_t(T)} = r_t \, dt - B_t(T)\,\sigma \, d\hat{W}_t$ $\quad \mathbb{Q}$-B.mtn.

$$\frac{dX_t}{X_t} = \underbrace{\left(- B_t(t_{k-1}) + B_t(t_k)\right)\sigma}_{\Sigma_t} \, d\hat{W}_t^{(k)}$$

$\mathbb{Q}_k$ - B.mtn

$$\left(d\hat{W}_t^{(k)} = B_t(t_k)\sigma \, dt + d\hat{W}_t \right)$$

$$g_t^{(k)} = P_t(t_k)\, \mathbb{E}^{\mathbb{Q}_k}\left[\left(X_{t_{k-1}} - \overset{(1+\Delta t_k K)}{\mathcal{K}}\right)_+ \right]$$

$$\Rightarrow \boxed{g_t^{(k)} = P_t(t_k)\left[X_t \, \Phi(d_+) - \mathcal{K} \, \Phi(d_-) \right]}$$

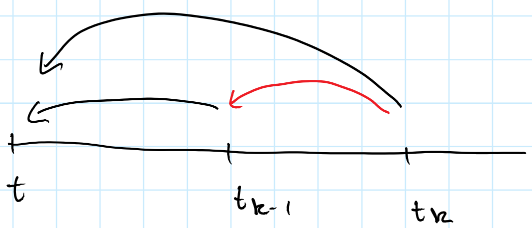
$$\Rightarrow \boxed{\begin{gathered} g_t^{(k)} = P_t(t_k)\left[X_t \, \Phi(d_+) - \mathcal{K} \, \Phi(d_-) \right] \\ d_\pm = \frac{\ln(X_t/\mathcal{K}) \pm \int_t^{t_{k-1}} \Sigma_s^2 \, ds}{\left(\int_t^{t_{k-1}} \Sigma_s^2 \, ds \right)^{1/2}} \end{gathered}}$$

$$\text{Cap} = \sum_{k=1}^{n} g_t^{(k)}$$

Black Implied Vols:

assume each caplet has a future LIBOR that is log-normally distributed

$$\ell_{t_{k-1}}(t_k) \overset{d}{=} \underbrace{\ell_t(t_{k-1}, t_k)}_{\substack{\text{today's Forward rate} \\ \text{For } t_{k-1} \to t_k \text{ period.}}} \cdot \overset{\text{GBM}}{Y_{t_{k-1}}}$$



$$P_t(t_k) = P_t(t_{k-1}) \left(1 + \Delta t_k \, \ell_t(t_{k-1}, t_k)\right)^{-1}$$

$$\frac{d\ell_t^{(k)}}{\ell_t^{(k)}} = \sigma \, \underbrace{dW_t}_{\text{"}\mathbb{Q}\text{"-B.mtn}}, \quad \text{starts at } \uparrow$$

$$\Rightarrow \ell_{t_{k-1}}^{(k)} = \ell_t(t_{k-1}, t_k) \, e^{-\frac{1}{2}\sigma^2 (t_{k-1} - t) + \sigma (W_{t_{k-1}} - W_t)}$$

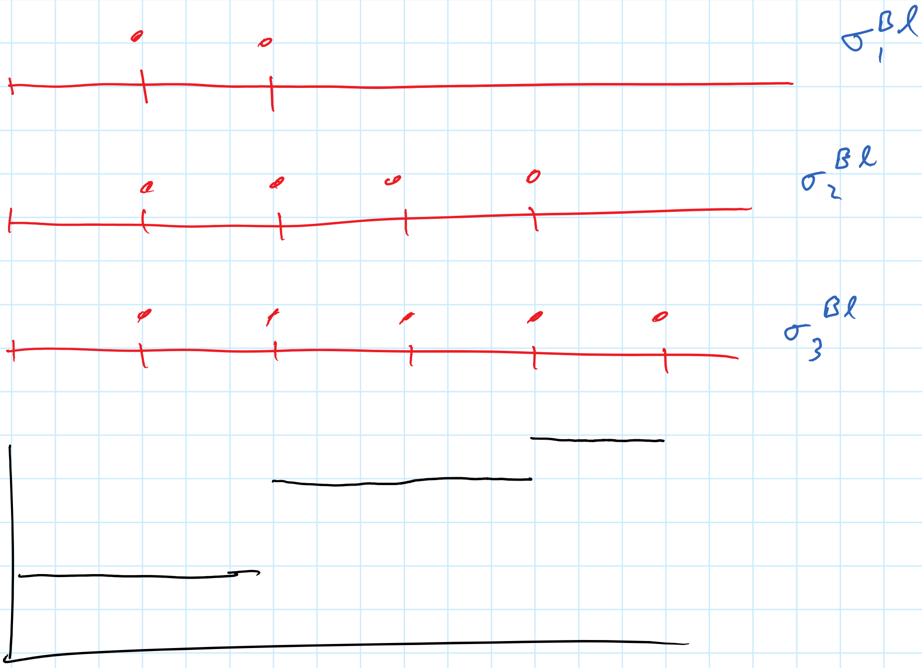
$$g_t^{B(k)} = P_t(t_k) \, \mathbb{E}^{\text{"}\mathbb{Q}\text{"}}\left[\left(\ell_{t_{k-1}}^{(k)} - K \right)_+ \Delta t_k \right]$$

$$\boxed{\begin{gathered} g_t^{B(k)} = P_t(t_k) \left\{ \ell_t(t_{k-1}, t_k) \, \Phi(d_+) - K \, \Phi(d_-) \right\} \Delta t_k \\ d_\pm = \frac{\ln\left(\ell_t(t_{k-1}, t_k)/K\right) \pm \sigma^2 (t_{k-1} - t)}{\sigma (t_{k-1} - t)^{1/2}} \end{gathered}}$$

$$\boxed{\check{g}_t^{B(k)} = P_t(t_k)\left\{ \ell_t(t_{k-1},t_k)\,\Phi(d_+) - K\,\Phi(d_-)\right\}\Delta t_k \qquad d_\pm = \frac{\ln\left(\ell_t(t_{k-1},t_k)/K\right) \pm \sigma^2 (t_{k-1}-t)}{\sigma\,(t_{k-1}-t)^{1/2}}}$$

$$\sum_{k=1}^{n} g_t^{(k)} = \sum_{k=1}^{n} g_t^{B(k)}(\sigma)$$

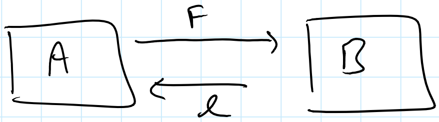
Find an implied vol



Swaptions

28 November 2013 19:29

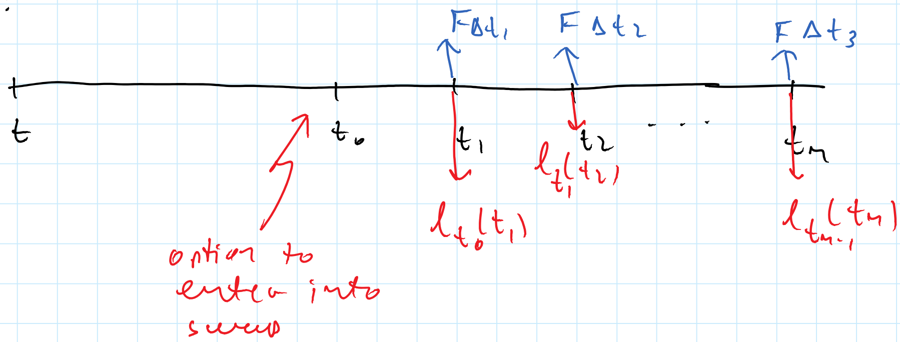
Swap Option (swaption)



payer swaption:

allows A to enter into the payer swap. (i.e. A pays fixed & receives float) at a future time t_0, and there is a collection of reset/payment dates (tenor)

A pays:



payoff at t_0: if $V^{Fl}_{t_0} > V^{Fixed}_{t_0}$, then

$$\mathcal{C}_{t_0} = V^{Fl}_{t_0} - V^{fixed}_{t_0}$$

otherwise 0.

$$\mathcal{C}_{t_0} = \left(V^{Fl}_{t_0} - V^{fixed}_{t_0} \right)_+$$

$$V^{fixed}_{t_0} = F \sum_{k=1}^{n} P_{t_0}(t_k)\, \Delta t_k$$

$$V^{Fl}_{t_0} = 1 - P_{t_0}(t_n)$$

so $$\mathcal{C}_{t_0} = \left(1 - P_{t_0}(t_n) - F \sum_{k=1}^{n} P_{t_0}(t_k)\, \Delta t_k \right)_+$$

in an affine model: $P_t(T) = e^{A_t(T) - B_t(T) r_t}$

Find r^* s.t.

Find r^* s.t.

$$\sum_{k=1}^{n} c_k P_{t_0}(t_k; r^*) = 1 \quad , \quad c_1 = F\Delta t_1,\ c_2 = F\Delta t_2, \ldots, c_{n-1} = F\Delta t_{n-1}, \quad c_n = 1 + \Delta t_n F$$

$$\mathcal{Q}_{t_0} = \left(1 - \sum_k c_k P_{t_0}(t_k)\right) \mathbb{1}_{r_{t_0} > r^*}$$

$$= \mathbb{1}_{r_{t_0} > r^*} - \sum_k c_k P_{t_0}(t_k)\, \mathbb{1}_{r_{t_0} > r^*}$$

$$\mathbb{1}_{r_{t_0} > r^*} = \mathbb{1}_{P_{t_0}(t_k) < P_{t_0}(t_k; r^*)}, \quad P_{t_0}(t_k; r^*) \longrightarrow P_k^*$$

$$\hookrightarrow \quad -B_{t_0}(t_k)\, r_{t_0} < -B_{t_0}(t_k)\, r^*$$

$$A_{t_0}(t_k) - B_{t_0}(t_k)\, r_0 < A_{t_0}(t_k) - B_{t_0}(t_k)\, r^*$$

$$\Rightarrow \quad \Rightarrow \quad \Leftarrow \quad \Leftarrow$$

$$g_t = \mathbb{E}^{\mathbb{Q}}_t\left[e^{-\int_t^{t_0} r_s\,ds} \left(\mathbb{1}_{r_{t_0} > r^*} - \sum_k c_k P_{t_0}(t_k)\, \mathbb{1}_{P_{t_0}(t_k) < P_k^*} \right)\right]$$

$$= \mathbb{E}^{\mathbb{Q}}_t\left[e^{-\int_t^{t_0} r_s\,ds}\, \mathbb{1}_{r_{t_0} > r^*}\right] \rightarrow \text{use } P_t(t_0) \text{ as numeraire.}$$

$$- \sum_k c_k \underbrace{\mathbb{E}^{\mathbb{Q}}_t\left[e^{-\int_t^{t_0} r_s\,ds}\, \boxed{P_{t_0}(t_k)}\, \mathbb{1}_{P_{t_0}(t_k) < P_k^*}\right]}_{h_t^k}$$

$$\frac{h_t^k}{P_t(t_k)} = \mathbb{E}^{\mathbb{Q}_k}_t\left[\mathbb{1}_{P_{t_0}(t_k) < P_k^*}\right]$$

$$\Rightarrow \quad h_t^k = P_t(t_k)\, \mathbb{Q}_k\left(P_{t_0}(t_k) < P_k^* \right)$$

$$\frac{dP_t(t_k)}{P_t(t_k)} = r_t\,dt - B_t(t_k)\,\sigma\, d\hat{W}_t$$

$$= \left(r_t + B_t^2(t_k)\,\sigma^2\right) dt - B_t(t_k)\,\sigma\, d\hat{W}_t^k$$

$$d\hat{W}_t^k = B_t(t_k)\,\sigma\, dt + d\hat{W}_t$$

$$\mathbb{Q}_k\left(P^{-1}_{t_0}(t_k) > P^{\alpha-1}_k \right)$$

$$X_t = \frac{P_t(t_0)}{P_t(t_n)} \xrightarrow[t \uparrow t_0]{} P^{-1}_{t_0}(t_n)$$

is a $\mathbb{Q}_k$-mtg ?

. . .

$$\mathcal{C}_{t_0} = \left(1 - P_{t_0}(t_M) - F \sum_k \Delta t_k P_{t_0}(t_k) \right)_+$$

$$X_t = \sum_k \Delta t_k P_t(t_k) \quad \text{(annuity)}$$

$$\frac{dX_t}{X_t} = r_t\, dt + (\sigma)\, d\hat{W}_t$$

$X > 0$ a.s.

use X as a numeraire

$$\frac{g_t}{X_t} = \mathbb{E}^{\mathbb{Q}_X}\left[\frac{\left(1 - P_{t_0}(t_M) - F \sum_k \Delta t_k P_{t_0}(t_k)\right)_+}{\sum_k \Delta t_k P_{t_0}(t_k)} \right]$$

$$= \mathbb{E}^{\mathbb{Q}_X}\left[\left(\frac{1 - P_{t_0}(t_M)}{\sum_k \Delta t\, P_{t_0}(t_k)} - F \right)_+ \right]$$

$\rightarrow$ fair swaprate @ t_0!

$$= \mathbb{E}^{\mathbb{Q}_X}\left[\left(S_{t_0} - F \right)_+ \right]$$

$$g_t = X_t\, \mathbb{E}^{\mathbb{Q}_X}\left[\left(S_{t_0} - F \right)_+ \right]$$

$$S_t = \frac{P_t(t_0) - P_t(t_M)}{\sum_k \Delta t_k P_t(t_k)}$$ is a $\mathbb{Q}^X$-mtg.

can compute $dS_t = (\ )dt + (\ )d\hat{W}_t$?

$= (0)dt + (\ )d\hat{W}^X_t$

L S M lognormal swaprate model

$$\frac{dS_t}{S_t} = \sigma\, d\hat{W}^X_t$$ $\longrightarrow$ get a Formula for swaption.

$$\boxed{\begin{aligned} g_t &= X_t\left(S_t\, \Phi(d_+) - F\, \Phi(d_-)\right) \\ d_\pm &= \frac{\ln(S_t/F) \pm \frac{1}{2}\sigma^2(t_0 - t)}{\sigma\,(t_0-t)^{1/2}} \end{aligned}}$$