Heston Model

13 November 2013 14:10

FTAP: no arb $\Longleftrightarrow$ $\exists$ $\mathbb{Q} \sim \mathbb{P}$ s.t all traded assets F, $\frac{F}{M}$ is a $\mathbb{Q}$-mtg

holds always!

even if market is incomplete.

complete markets have unique $\mathbb{Q}$!

incomplete markets do not have unique $\mathbb{Q}$!

$$dX_t = \mu_t \, dt + \sigma_t \, dW_t \quad \text{\(\mathbb{P}\)-B.mtn}$$

$$d\hat{W}_t = \lambda_t \, dt + dW_t$$

$$\frac{f_t}{M_t} = \mathbb{E}^{\mathbb{Q}}\left[\frac{f_T}{M_T} \right]$$

$$dX_t = (\underbrace{\mu_t - \lambda_t}_{\text{m.p.r.}} \sigma_t) \, dt + \sigma_t \, d\hat{W}_t \quad \text{\(\mathbb{Q}\)-B.mtn.}$$

when X is traded then $\mu_t - \lambda_t \sigma_t = r X_t$

$$\frac{dF_t}{F_t} = \sigma \, \underbrace{dW_t}_{\mathbb{Q}} \qquad \text{Black model of Futures prices.}$$

$$\frac{dF_t}{F_t} = \mu_t \, dt + \sigma \, dW_t^{\mathbb{P}}$$

want to correct for stochastic volatility

$$\frac{dF_t}{F_t} = \mu_t \, dt + \gamma \sqrt{v_t} \, dW_t^{\mathbb{P},F}$$

$$dv_t = \kappa^{\mathbb{P}} (\theta^{\mathbb{P}} - v_t) \, dt + \alpha \sqrt{v_t} \, dW_t^{\mathbb{P},v}$$

$W_t^{\mathbb{P},F}$ & $W_t^{\mathbb{P},v}$ are correlated (ρ)

$\kappa^{\mathbb{P}}$: m.r. rate

$\theta^{\mathbb{P}}$: m.r. level

α: vol-vol

m.r. rate m.r. level vol-vol

such processes are called Feller processes.
($2\kappa^{\mathbb{P}}\theta^{\mathbb{P}} > \alpha^2$ to avoid v from hitting zero)

$$\frac{dF_t}{F_t} = \gamma\sqrt{v_t}\, dW_t^F$$

$\mathbb{Q}$ - B.mtn. corr. (ρ)

$$dv_t = \underbrace{\left(\kappa^{\mathbb{P}}(\theta^{\mathbb{P}} - v_t) - \alpha\lambda_t^v\sqrt{v_t}\right)}_{\mu_t^v} dt + \alpha\sqrt{v_t}\, dW_t^v$$

$$dW_t^F = \frac{\mu_t}{\gamma\sqrt{v_t}}\, dt + dW_t^{\mathbb{P},F}$$

$$\text{\&} \quad dW_t^v = \lambda_t^v\, dt + dW_t^{\mathbb{P},v}$$

choices of λ_t^v:

1) $\lambda_t^v = c\sqrt{v_t}$

$$\Rightarrow \mu_t^v = \kappa^{\mathbb{P}}\theta^{\mathbb{P}} - \underbrace{(\kappa^{\mathbb{P}} + \alpha c)}_{\kappa} v_t$$

$$= \kappa\left(\frac{\kappa^{\mathbb{P}}\theta^{\mathbb{P}}}{\kappa} - v_t\right)$$

2) $\lambda_t^v = \frac{\ell}{\sqrt{v_t}} + c\sqrt{v_t}$

$$\mu_t^v = \kappa(\theta - v_t)$$

$$\kappa = \ldots, \quad \theta = \ldots$$

under #2 m.p.r.

$$\boxed{\begin{aligned} \frac{dF_t}{F_t} &= \gamma\sqrt{v_t}\, dW_t^F \\ dv_t &= \kappa(\theta - v_t)\, dt + \alpha\sqrt{v_t}\, dW_t^v \end{aligned}}$$

$$\frac{dF_t}{F_t} = \gamma \sqrt{v_t}\, dW_t^F$$

$$dv_t = \kappa(\theta - v_t)\, dt + \alpha \sqrt{v_t}\, dW_t^v$$

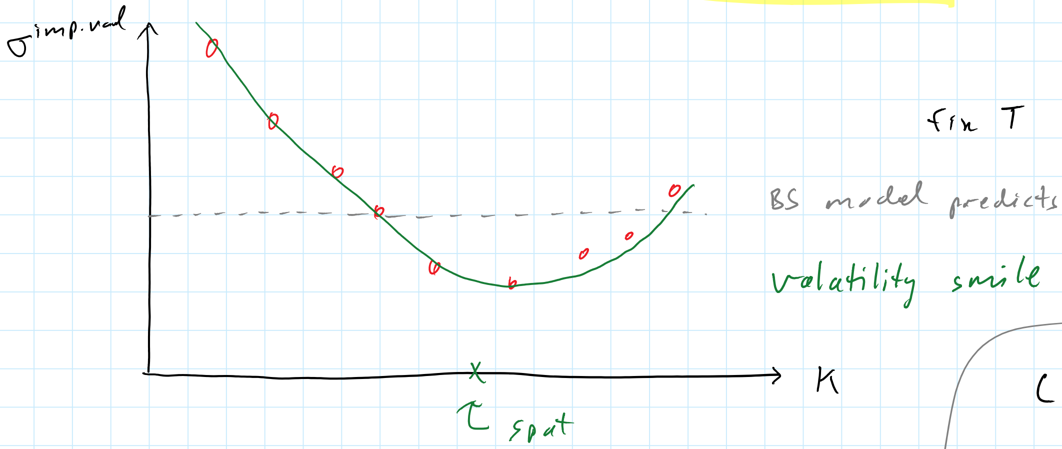
Heston model

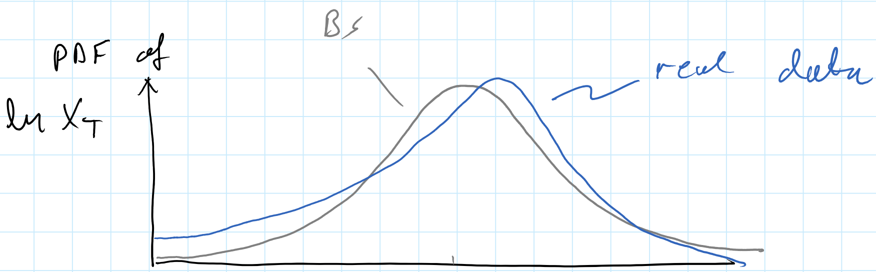
$$\frac{dF_t}{F_t} = \gamma \sqrt{v_t}\, dW_t^F$$

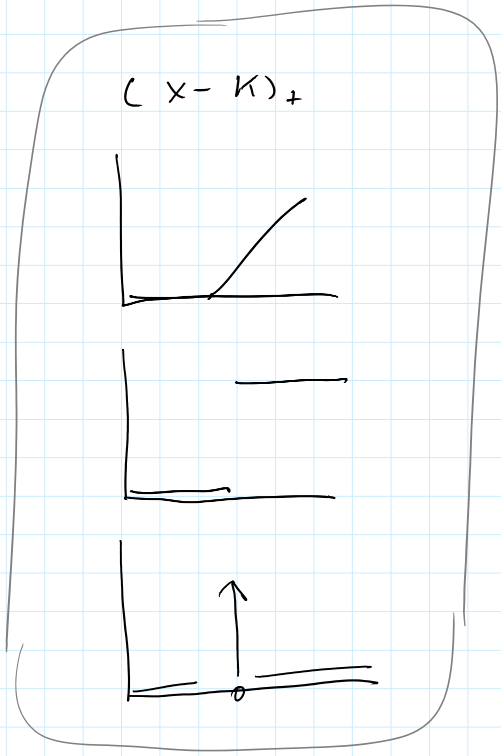
$$dv_t = \kappa(\theta - v_t)\, dt + \alpha \sqrt{v_t}\, dW_t^v$$

implied volatility:

$$V^*(K,T) = V^{BS}(r, \sigma^{imp.vol}(K,T); K, T)$$







can simulate F, v:

1) $X_t = \ln F_t$

$$dX_t = -\frac{1}{2}\gamma^2 v_t\,dt + \gamma\sqrt{v_t}\,dW_t^F$$

$$,\quad dv_t = \kappa(\theta - v_t)\,dt + \alpha\sqrt{v_t}\,dW_t^v$$

$$X_{t_n} - X_{t_{n-1}} = -\frac{1}{2}\gamma^2\, v_{t_{n-1}}^+\,\Delta t + \alpha\sqrt{v_{t_{n-1}}^+}\,\sqrt{\Delta t}\;Z_{1,n}$$

$$v_{t_n} - v_{t_{n-1}} = \kappa(\theta - v_{t_{n-1}}^+)\,\Delta t + \alpha\sqrt{v_{t_{n-1}}^+}\,\sqrt{\Delta t}\left(\rho Z_{1,n} + \sqrt{1-\rho^2}\,Z_{2,n}\right)$$

$Z_{1,\cdot}$, $Z_{2,\cdot}$ are iid $N(0,1)$

$$v_\circ^+ = \max(v_\circ, 0)$$

$$g_t = \mathbb{E}_t^{Q}[\,(e^{X_T} - K)_+\,] = g(t, X_t, v_t)$$

$$\begin{aligned} dg_t = {} & \partial_t g_t\,dt + \partial_x g_t\,dX_t + \partial_v g_t\,dv_t \\ & + \tfrac{1}{2}\partial_{xx} g_t\,(dX_t)^2 + \tfrac{1}{2}\partial_{vv} g_t\,(dv_t)^2 \\ & + \partial_{xv} g_t\,(dX_t\,dv_t) \end{aligned}$$

$$\begin{aligned} = dt\,\{ & \partial_t g_t + (-\tfrac{1}{2}\gamma^2 v_t)\,\partial_x g_t + \kappa(\theta - v_t)\,\partial_v g_t \\ & + \tfrac{1}{2}\partial_{xx} g_t\,\gamma^2 v_t + \tfrac{1}{2}\partial_{vv} g_t\,\alpha^2 v_t \\ & + \partial_{xv} g_t\,\rho\alpha\gamma v_t\,\} \\ & + \cdots \end{aligned}$$

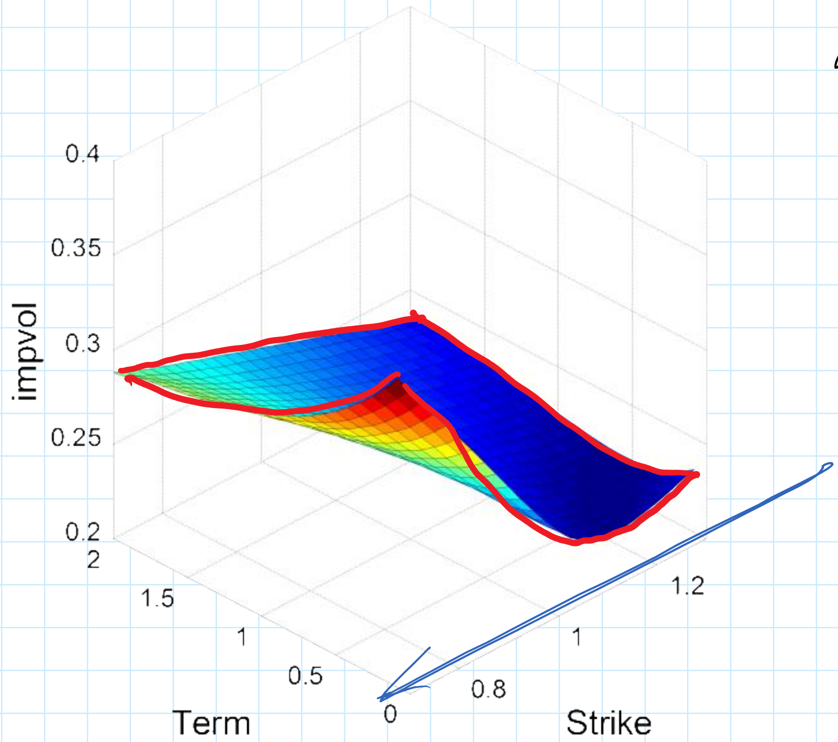
$\{\ \} = 0$ since g_t is a mtg and holds true $\forall$ paths & all starting points

$$\Rightarrow \boxed{\begin{cases} \partial_t g - \frac{1}{2}\gamma^2 v\,\partial_x g + \frac{1}{2}\gamma^2 v\,\partial_{xx} g \\ \quad + \kappa(\theta - v)\,\partial_v g + \frac{1}{2}\alpha^2 v\,\partial_{vv} g + \rho\alpha\gamma v\,\partial_{xv} g = 0 \\ \qquad g(T, x, v) = (e^x - K)_+ \end{cases}}$$

can be solved analytically.

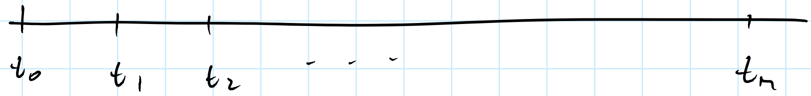
Sample Implied Vol

13 November 2013 16:16



$$dv_t = \kappa(\theta - v_t)\,dt + \alpha\sqrt{v_t}\,dW_t^v$$

Variance Swaps:



$$\text{"realized variance"} \triangleq \sum_{k=1}^{N} \left(\frac{F_{t_k}}{F_{t_{k-1}}} - 1 \right)^2 = R$$

pays $(R-K)$ @ T.

$$\frac{F_{t_k}}{F_{t_{k-1}}} - 1 \cong \frac{dF_t}{F_t}$$

$$R \cong \sum_{k=1}^{N} \left(\frac{dF_{t_k}}{F_{t_k}}\right)^2 \qquad \text{now} \quad \frac{dF_t}{F_t} = \gamma \sqrt{v_t}\, dW_t$$

$$= \sum_{k=1}^{N} \left(\gamma \sqrt{v_{t_{k-1}}}\right)^2 \left(W_{t_k} - W_{t_{k-1}}\right)^2$$

$$\mathbb{E}[R] \xrightarrow[N\to+\infty]{} \mathbb{E}\left[\int_0^T \gamma^2 v_s \, ds\right]$$

$$= \gamma^2 \int_0^T \mathbb{E}[v_s] \, ds$$

$$dv_t = \kappa(\theta - v_t)\, dt + \alpha \sqrt{v_t}\, dW_t^v$$

$$v_t - v_0 = \int_0^t \kappa(\theta - v_s)\, ds + \alpha \int_0^t \sqrt{v_s}\, dW_s^v$$

$$\Rightarrow \mathbb{E}[v_t] - v_0 = \int_0^t \kappa(\theta - \mathbb{E}[v_s])\, ds + 0$$

let $m_t \stackrel{\Delta}{=} \mathbb{E}[v_t]$

$$m_t - m_0 = \int_0^t \kappa(\theta - m_s)\, ds$$

$$\begin{cases} \dot{m}_t = \kappa(\theta - m_t) \\ m_0 = v_0 \end{cases}$$

$$m_t = v_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t})$$

Fair variance strike K is s.t. $\mathbb{E}[R - K] = 0$.

pays $(R_T - K)_+$ @ T

assume: $\dfrac{dR_t}{R_t} = \sigma(K,T)\, dW_t$

$R_0 = \hat{V}_0(T)$ — Fair variance swap strike of T-mat.