One-Period Binomial

11 September 2013 14:16

$p \in (0,1)$

$$\begin{tikzcd}[row sep=small] & A_u \\ A_0 \arrow[ur, "p"] \arrow[dr] & \\ & A_d \end{tikzcd}$$

$$A_0 = \frac{p A_u + (1-p) A_d}{1+r}$$

$$\begin{tikzcd}[row sep=small] & B_u \\ B_0 \arrow[ur] \arrow[dr] & \\ & B_d \end{tikzcd}$$

$$A_1 = A_u x_1 + A_d (1 - x_1)$$

$$x_1 = \underset{p}{1}, \underset{(1-p)}{0}$$

$$B_1 = B_u x_1 + B_d (1 - x_1)$$

$$\begin{tikzcd}[row sep=small] & A_u \\ A_0 \arrow[ur] \arrow[dr] & \\ & A_d \end{tikzcd} \qquad \begin{tikzcd}[row sep=small] & B_u \\ B_0 \arrow[ur] \arrow[dr] & \\ & B_d \end{tikzcd} \qquad \left(\frac{A_u}{B_u} > \frac{A_d}{B_d} \right)$$

α $\qquad$ β

$B_0 \neq 0$

$B_u \neq 0$

$B_d \neq 0$

$$V_0 = \alpha A_0 + \beta B_0$$

$$V_1 = \alpha A_1 + \beta B_1$$

an arbitrage is a strategy s.t.

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i) $V_0 = 0$

ii) $\exists\, t$ s.t. a) $\mathbb{P}(V_t \geq 0) = 1$

b) $\mathbb{P}(V_t > 0) > 0$

$V_0 = 0 \Rightarrow \alpha A_0 + \beta B_0 = 0$

$\Rightarrow \beta = -\alpha \frac{A_0}{B_0}$

$$0 \begin{cases} \alpha A_u + \beta B_u = \alpha\left(A_u - \frac{A_0}{B_0} B_u\right) \\ \alpha A_d + \beta B_d = \alpha\left(A_d - \frac{A_0}{B_0} B_d\right) \end{cases}$$

$$\left.\begin{array}{l} A_u - \frac{A_0}{B_0} B_u > 0 \\ \& \; A_d - \frac{A_0}{B_0} B_d < 0 \end{array}\right\} \text{all other cases lead to arb.}$$

$$\boxed{\frac{A_d}{B_d} < \frac{A_0}{B_0} < \frac{A_u}{B_u}}$$ no arbitrage condition

$\Longleftrightarrow$

$\exists$ some $q \in (0,1)$ s.t.

$$\boxed{\frac{A_0}{B_0} = q \frac{A_u}{B_u} + (1-q) \frac{A_d}{B_d}}$$

$\Longleftrightarrow$ no arbitrage!

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no arbitrage iff $\exists \; q \in (0,1)$ s.t.

$$\tilde{A}_0 = \mathbb{E}^{\mathbb{Q}}[\tilde{A}_1]$$

where $\tilde{A}_t = \dfrac{A_t}{B_t}$ (relative price)

-1 $\qquad\qquad$ $+5$ $\qquad\qquad$ 14×5

$$10 \left< \begin{matrix} 5 \\ \\ 20 \end{matrix} \right. \qquad 2 \left< \begin{matrix} 15 \\ \\ 4 \end{matrix} \right. \qquad 0 \left< \begin{matrix} 14 \times 5 \\ \\ 0 \end{matrix} \right.$$

$14 \times 5 \times 0$ $\leftarrow q$

$+ \; 0 \times 1$ $\leftarrow 1-q$

$$5 = \frac{10}{2} \left< \begin{matrix} 5/15 = \frac{1}{3} \\ q \\ 20/4 = 5 \end{matrix} \right.$$

A Third Asset

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$$
\begin{tikzcd}[row sep=small]
 & A_u \\
A_0 \arrow[ur] \arrow[dr] & \\
 & A_d
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=small]
 & B_u \\
B_0 \arrow[ur] \arrow[dr] & \\
 & B_d
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=small]
 & C_u \\
C_0 \arrow[ur] \arrow[dr] & \\
 & C_d
\end{tikzcd}
$$

$$\tilde{A}_0^B = \mathbb{E}^{\mathbb{Q}^{BA}}\left[\tilde{A}_1^B\right] \rightarrow q^{BA} \in (0,1)$$

$$\tilde{C}_0^B = \mathbb{E}^{\mathbb{Q}^{BC}}\left[\tilde{C}_1^B\right] \rightarrow q^{BC} \in (0,1)$$

suppose we have q^{BA}, then compute

$$\frac{\hat{C}_0}{B_0} \stackrel{?}{=} \frac{C_u}{B_u} \cdot q^{BA} + \frac{C_d}{B_d}(1-q^{BA}) > \frac{C_0}{B_0}$$

$$
\begin{tikzcd}[row sep=small]
 & 20 \\
10 \arrow[ur] \arrow[dr] & \\
 & 5
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=small]
 & 1 \\
1 \arrow[ur] \arrow[dr] & \\
 & 1
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=small]
 & 120 \\
110 \arrow[ur] \arrow[dr] & \\
 & 100
\end{tikzcd}
$$

α $\qquad$ β $\qquad$ -1

$$10 = 20\, q^{BA} + 5(1-q^{BA})$$

$$\Rightarrow \quad q^{BA} = \frac{1}{3}$$

$$110 = 120\, q^{BC} + 100(1-q^{BC})$$

$$\Rightarrow q^{BC} = \frac{10}{20} = \frac{1}{2}$$

$$\hat{C}_0 = 120\,\frac{1}{3} + 100 \times \frac{2}{3} = 40 + 66\tfrac{2}{3} < 110$$

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$$V_0 = 10\alpha + \beta - 110 = 0$$

$$\Rightarrow \quad \beta = 110 - 10\alpha = 90$$

$$0 \left\langle \begin{array}{l} 20\alpha + \beta - 120 = 10\alpha - 10 = 10 \\ \\ 5\alpha + \beta - 100 = -5\alpha + 10 = 0 \end{array} \right.$$

$\alpha = 2$

Replication

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$$A_0 \begin{cases} A_u \\ A_d \end{cases} \qquad B_0 \begin{cases} B_u \\ B_d \end{cases} \qquad C_0 \begin{cases} C_u = \alpha A_u + \beta B_u \\ C_d = \alpha A_d + \beta B_d \end{cases}$$

$\alpha \qquad\qquad \beta$

choose (α, β) s.t. $\alpha A_1 + \beta B_1 = C_1$ $\mathbb{P}$-a.s.

$\Rightarrow \quad C_0 = \alpha A_0 + \beta B_0$!

otherwise $\exists$ an arb!

you will find that

$$\boxed{\frac{C_0}{B_0} = q \frac{C_u}{B_u} + (1-q) \frac{C_d}{B_d}}$$

where $q = \dfrac{\frac{A_0}{B_0} - \frac{A_d}{B_d}}{\frac{A_u}{B_u} - \frac{A_d}{B_d}}$

moreover q satisfies

$$\frac{A_0}{B_0} = q \frac{A_u}{B_u} + (1-q) \frac{A_d}{B_d}$$

no arb $\Longleftrightarrow$ $\exists\, \mathbb{Q} \sim \mathbb{P}$ s.t. for all traded assets X, we have and traded assets X, we have

$$\tilde{X}_0 = \mathbb{E}^{\mathbb{Q}}[\tilde{X}_1]$$

where $\tilde{X}_t = \dfrac{X_t}{B_t}$ and

$\mathbb{P}(B_t > 0) = 1$ (B is called a numeraire asset)

Multi-Period Binomial

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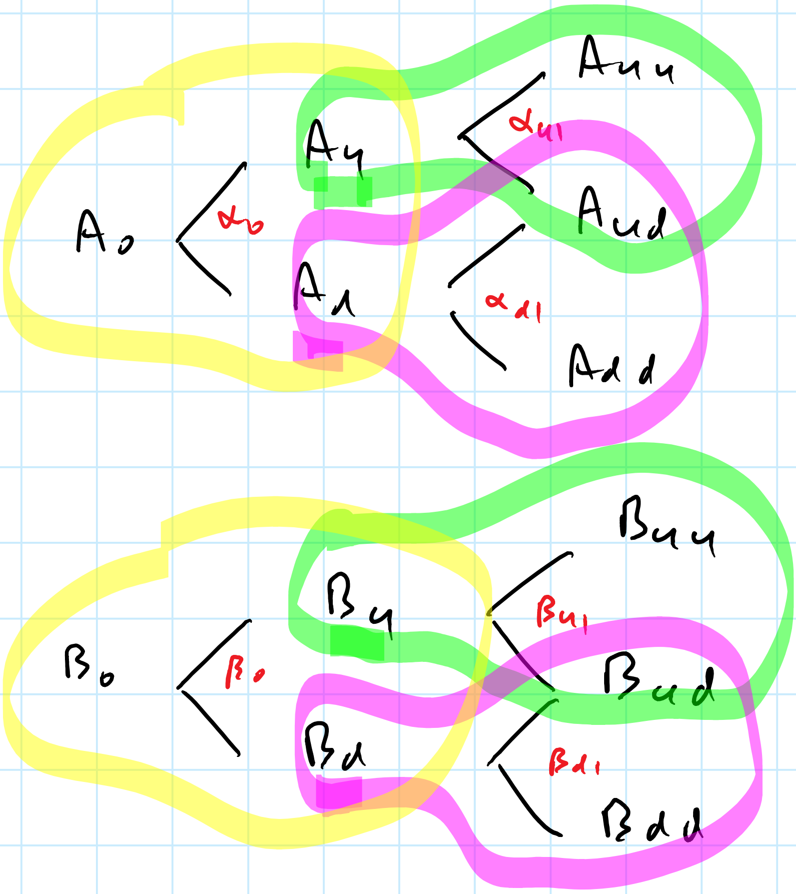
$$A_n = A_{n-1}\, e^{c\, x_n}$$

$x_1, x_2, \ldots$ iid Bernoulli ± 1 with p, $1-p$



an arbitrage is strategy that is adapted to the filtration generated by asset prices ($\alpha_t \in \mathcal{F}_t$) s.t.

i) $V_0 = 0$

ii) $\exists\, t$ s.t.

a) $\mathbb{P}(V_t \geq 0) = 1$

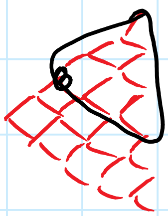
b) $\mathbb{P}(V_t > 0) > 0$

martingale measure

no arb $\iff \exists\, \mathbb{Q} \sim \mathbb{P}$ s.t. $\forall\, t < s$ and all traded assets X, we have

$$\boxed{\tilde{X}_t = \mathbb{E}^{\mathbb{Q}}[\tilde{X}_s \mid \mathcal{F}_t]}$$

i.e. relative prices are martingale!



if B is the bank account, then $\mathbb{Q}$ is called the risk-neutral measure

Matching Drift & Volatility

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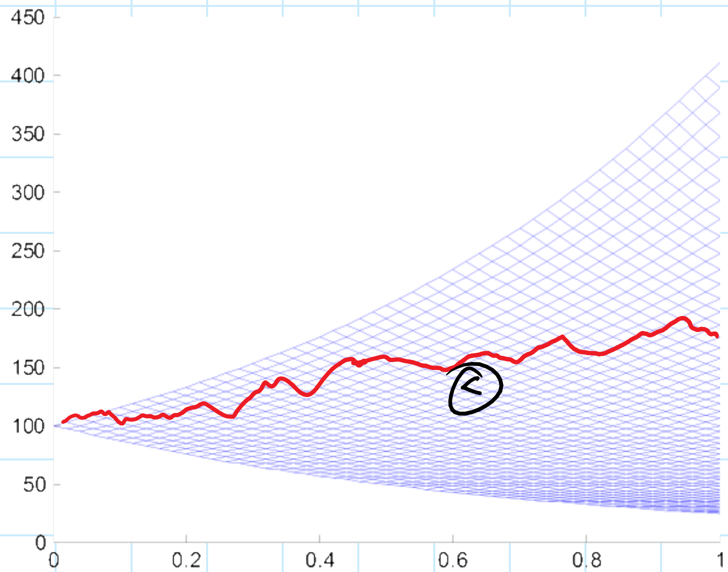
$$A_n = A_{n-1}\, e^{c\, x_n}, \qquad c = \sigma\sqrt{\Delta t}$$



$$\ln(A_T/A_0) = \sigma\sqrt{\Delta t}\sum_{n=1}^{N} x_n$$

$\hookrightarrow N\Delta t = T$

$$\mathbb{E}\left[\ln\left(\frac{A_T}{A_0}\right)\right] = \sigma\sqrt{\Delta t}\sum_{n=1}^{N}\mathbb{E}[x_n]$$

$$= \sigma\sqrt{\Delta t}\, N\,.\,(2p-1) = (\mu - \tfrac{1}{2}\sigma^2)\,T$$

$$\Rightarrow \quad 2p-1 = \frac{\mu - \frac{1}{2}\sigma^2}{\sigma}\,.\,\sqrt{\Delta t}$$

$$\boxed{p = \frac{1}{2}\left[1 + \frac{\mu - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right]}$$

$$\mathbb{V}\left[\ln\left(\frac{A_T}{A_0}\right)\right] = \sigma^2 \, \Delta t \cdot N \cdot \mathbb{V}[x_1]$$

$$\mathbb{V}[x_1] = \mathbb{E}[x_1^2] - \left(\mathbb{E}[x_1]\right)^2$$

$$= 1 - \left(\left(\frac{\mu - \frac{1}{2}\sigma^2}{\sigma}\right)\sqrt{\Delta t}\right)^2$$

$$\Rightarrow \mathbb{V}\left[\ln \frac{A_T}{A_0}\right] = \sigma^2 \, T \cdot \left(1 - \left(\frac{\mu - \frac{1}{2}\sigma^2}{\sigma}\right)^2 \Delta t\right)$$

$$B_t = e^{rt} \quad , \qquad B_n = B_{n-1} \, e^{r \Delta t}$$

$$\frac{A_t}{B_t} = \mathbb{E}^{\mathbb{Q}}\left[\frac{A_s}{B_s} \,\middle|\, \mathcal{F}_t\right]$$

$$A_{n-1} \begin{cases} A_{n-1} \, e^{\sigma\sqrt{\Delta t}} \\ A_{n-1} \, e^{-\sigma\sqrt{\Delta t}} \end{cases} \qquad\qquad B_{n-1} \begin{cases} B_{n-1} \, e^{r\Delta t} \\ B_{n-1} \, e^{r\Delta t} \end{cases}$$

$$\frac{A_{n-1}}{B_{n-1}} = \frac{A_{n-1}}{B_{n-1}} \, e^{\sigma\sqrt{\Delta t} - r\Delta t} \, q + \frac{A_{n-1}}{B_{n-1}} \, e^{-\sigma\sqrt{\Delta t} - r\Delta t} \, (1-q)$$

$$\Rightarrow \quad q = \frac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}$$

$$e^{x} \sim 1 + x + \frac{1}{2}x^2 + o(x^2)$$

$$e^x \sim 1 + x + \tfrac{1}{2}x^2 + o(x^2)$$

$$\text{num} = (1 + r\Delta t \qquad\qquad) - (1 - \sigma\sqrt{\Delta t} + \tfrac{1}{2}\sigma^2 \Delta t) + \cdots$$

$$= \sigma\sqrt{\Delta t} + (r - \tfrac{1}{2}\sigma^2)\Delta t + \cdots$$

$$\text{denom} = (1 + \sigma\sqrt{\Delta t} + \tfrac{1}{2}\sigma^2 \Delta t) - (1 - \sigma\sqrt{\Delta t} + \tfrac{1}{2}\sigma^2 \Delta t) + \cdots$$

$$= 2\sigma\sqrt{\Delta t} + \cdots$$

$$q = \frac{\text{num}}{\text{denom}} = \frac{1}{2}\left[1 + \frac{r - \tfrac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right] + \cdots$$

$$\mathbb{E}^{\mathbb{Q}}\left[\ln \frac{A_T}{A_0}\right] = (r - \tfrac{1}{2}\sigma^2)\, T$$

$$\mathbb{V}^{\mathbb{Q}}\left[\ln \frac{A_T}{A_0}\right] = \sigma^2 T + \xi \Delta t$$

Limiting Distribution

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$$\ln\left(\frac{A_T}{A_0}\right) = \sigma\sqrt{\Delta t}\sum_{n=1}^{N} x_n \xrightarrow[N\to+\infty]{d} \mathcal{N}(\cdot\,;\,\cdot)$$

$$\text{mean} = (\mu - \tfrac{1}{2}\sigma^2)T$$

$$\text{var} = \sigma^2 T$$

$$\iff A_T \overset{d}{=} A_0\, e^{(\mu-\frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\,Z}$$

$$Z \underset{\mathbb{P}}{\sim} \mathcal{N}(0;1)$$

$$\mathbb{E}[A_T] = A_0\, e^{(\mu-\frac{1}{2}\sigma^2)T}\,\mathbb{E}[e^{\sigma\sqrt{T}\,Z}]$$

$$\hookrightarrow e^{\frac{1}{2}\sigma^2 T}$$

Existence of Martingale Measure => No Arbitrage

11 September 2013 21:16

$A_t^{(1)}, A_t^{(2)}, \ldots, A_t^{(n)}$ non-negative $\mathbb{P}$-a.s.

B_t strictly positive $\mathbb{P}$-a.s.

$t = \{0, 1\}$

suppose $\exists\ \mathbb{Q} \sim \mathbb{P}$ s.t.

$$\frac{A_0^{(k)}}{B_0} = \mathbb{E}^{\mathbb{Q}}\left[\frac{A_1^{(k)}}{B_1}\right]$$

take a portfolio β of B
$\alpha^{(k)}$ of $A^{(k)}$

s.t. $\mathbb{P}(V_1 \geq 0) = 1$
$\mathbb{P}(V_1 > 0) > 0$

note: i) $\mathbb{E}[V_1] > 0$

$$V_0 = \beta B_0 + \alpha \cdot A_0$$

$$= B_0\left[\beta + \sum_{k=1}^{n} \alpha^{(k)} \frac{A_0^{(k)}}{B_0}\right] \qquad \frac{A_0^{(k)}}{B_0} = \mathbb{E}^{\mathbb{Q}}\left[\frac{A_1^{(k)}}{B_1}\right]$$

$$= B_0\, \mathbb{E}^{\mathbb{Q}}\left[\frac{B_1}{B_1}\beta + \sum_{k=1}^{n} \alpha^{(k)} \frac{A_1^{(k)}}{B_1}\right]$$

$$= B_0 \mathbb{E}\left[\frac{B_1}{B_1} \beta + \sum_{k=1} \alpha^{(k)} \frac{A_1^{(k)}}{B_1} \right]$$

$$= B_0 \mathbb{E}^{\bar{Q}}\left[\frac{\beta B_1 + \sum_k \alpha^{(k)} A_1^{(k)}}{B_1} \right]$$

$$= B_0 \mathbb{E}^{\bar{Q}}\left[\frac{V_1}{B_1} \right] > 0$$

CLT

17 September 2013 18:39

let $X = \sigma\sqrt{\Delta t}\sum_{n=1}^{N} x_i$, $\Delta t = \frac{T}{N}$

$$\mathbb{P}(x_i = +1) = p = \tfrac{1}{2}\left(1 - \frac{\mu - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right)$$

$$\mathbb{P}(x_i = -1) = 1-p = \tfrac{1}{2}\left(1 + \frac{\mu - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right)$$

let $\hat{\mu} \triangleq (\mu - \frac{1}{2}\sigma^2)$

Want to show that

$$X \xrightarrow[N \to +\infty]{d} \mathcal{N}(\hat{\mu}T, \sigma^2 T).$$

Do this via characteristic fn.

$$\mathbb{E}[e^{iuX}]$$

$$= \mathbb{E}[e^{iu\sigma\sqrt{\Delta t}\sum_{n=1}^{N} x_n}]$$

$$= \left(\mathbb{E}[e^{iu\sigma\sqrt{\Delta t}\, x_1}]\right)^N$$

now, $\mathbb{E}[e^{iu\sigma\sqrt{\Delta t}\, x_1}]$

$$= \mathbb{E}[1 + iu\sigma\sqrt{\Delta t}\, x_1 - \tfrac{1}{2}u^2\sigma^2\Delta t\, x_1^2 + o(\Delta t)]$$

$$= 1 + iu\sigma\sqrt{\Delta t}\,\mathbb{E}[x_1] - \tfrac{1}{2}u^2\sigma^2\Delta t\,\mathbb{E}[x_1^2] + o(\Delta t)$$

next, $\mathbb{E}[x_1] = 2p - 1 = \frac{\hat{\mu}}{\sigma}\sqrt{\Delta t}$

and $\mathbb{E}[x_1^2] = 1$

$$\Rightarrow \boxed{\begin{aligned}&\mathbb{E}[\,e^{iu\sigma\sqrt{\Delta t}\,x_1}\,] \\ &= 1 + iu\hat{\mu}\Delta t - \tfrac{1}{2}u^2\sigma^2\Delta t + o(\Delta t)\end{aligned}}$$

So then,

$$\mathbb{E}[\,e^{iuX}\,] = \Big(1 + (i\hat{\mu}u - \tfrac{1}{2}\sigma^2u^2)\overset{T/N}{\Delta t} + o(\Delta t)\Big)^N$$

$$\xrightarrow[N\to+\infty]{} e^{(i\hat{\mu}u - \frac{1}{2}\sigma^2u^2)T}$$

(since $(1 + \frac{a}{N})^N \xrightarrow[N\to+\infty]{} e^a$)

$$\therefore \boxed{X \xrightarrow[N\to+\infty]{d} \mathcal{N}(\hat{\mu}T;\ \sigma^2 T)}$$