Dynamic Hedging / PDE / Feynman-Kac / Measure Change

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- $dx_t = \mu_t^x\, dt + \sigma_t^x\, dW_t$ underlying risk

- trade
  - Bank act $dB_t = r_t B_t\, dt$

    $\hookrightarrow r_t = r(t, x_t)$

  - claim on x_t, $F_t = F(t, x_t)$

$$\frac{dF_t}{F_t} = \mu_t^F\, dt + \sigma_t^F\, dW_t$$

- value an option $g_T = \varphi(x_T)$

$$g_t = g(t, x_t)$$

$$\frac{dg_t}{g_t} = \mu_t^g\, dt + \sigma_t^g\, dW_t$$

$$\boxed{\frac{\mu_t^g - r_t}{\sigma_t^g} = \lambda_t = \lambda(t, x_t)} \iff \text{no arb.}$$

$$\begin{cases} \partial_t g + \left(\mu^x(t,x) - \lambda(t,x)\,\sigma^x(t,x)\right)\partial_x g \\ \qquad + \frac{1}{2}\left(\sigma(t,x)\right)^2 \partial_{xx} g = r(t,x)\, g \\ \qquad\qquad g(T,x) = \varphi(x) \end{cases}$$

$$\Leftrightarrow \quad g(t,x) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s,x_s)\,ds}\, \varphi(x_T) \,\middle|\, x_t = x \right]$$

$$dx_t = \left(\mu^x_t - \lambda_t \sigma^x_t\right) dt + \sigma^x_t\, d\hat{W}_t$$

$\mathbb{Q}$ - B.mtn

$$d\hat{W}_t = \lambda_t\, dt + dW_t$$

$$\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right) = \exp\left\{ -\frac{1}{2}\int_0^\infty \lambda_s^2\, ds - \int_0^\infty \lambda_s\, dW_s \right\}$$

for IR models choose $r(t,x) = x$

$$g(t,r) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_s\, ds}\, \varphi(r_T) \,\middle|\, r_t = r \right]$$

$$dr_t = \mu^r_t\, dt + \sigma^r_t\, dW_t$$

$$= \left(\mu^r - \lambda\, \sigma^r\right) dt + \sigma^r\, d\hat{W}$$

$$= (\mu_t^r - \lambda_t \sigma_t^r)\, dt + \sigma_t^r \, d\hat{W}_t$$

Vasicek Model I: Explicit Distribution

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Vasicek Model:

$$dr_t = \kappa(\theta - r_t)\,dt + \sigma\,dW_t \qquad \text{(Ornstein-Uhlenbeck)}$$

$$= [\underbrace{\kappa(\theta - r_t) - \lambda_t \sigma}_{(\kappa\theta + a) - (\kappa - b) r_t}]\,dt + \sigma\,d\hat{W}_t$$

$$\lambda_t = \frac{a + b r_t}{\sigma}$$

$$= \underbrace{(\kappa - b)}_{\hat{\kappa}} \left(\underbrace{\frac{\kappa\theta + a}{\kappa - b}}_{\hat{\theta}} - r_t \right)$$

so $$dr_t = \hat{\kappa}(\hat{\theta} - r_t)\,dt + \sigma\,d\hat{W}_t$$

$$x_t = e^{\hat{\kappa} t} r_t$$

$$dx_t = \kappa e^{\hat{\kappa} t} r_t\,dt + e^{\hat{\kappa} t} dr_t + d[e^{\kappa t}, r_t] \quad \to 0$$

$$= e^{\hat{\kappa} t}[\hat{\kappa} r_t\,dt + \hat{\kappa}(\hat{\theta} - r_t)\,dt + \sigma\,d\hat{W}_t]$$

$$= e^{\hat{\kappa} t}[\hat{\kappa}\hat{\theta}\,dt + \sigma\,d\hat{W}_t]$$

$e^{\hat{\kappa} t} r_t$

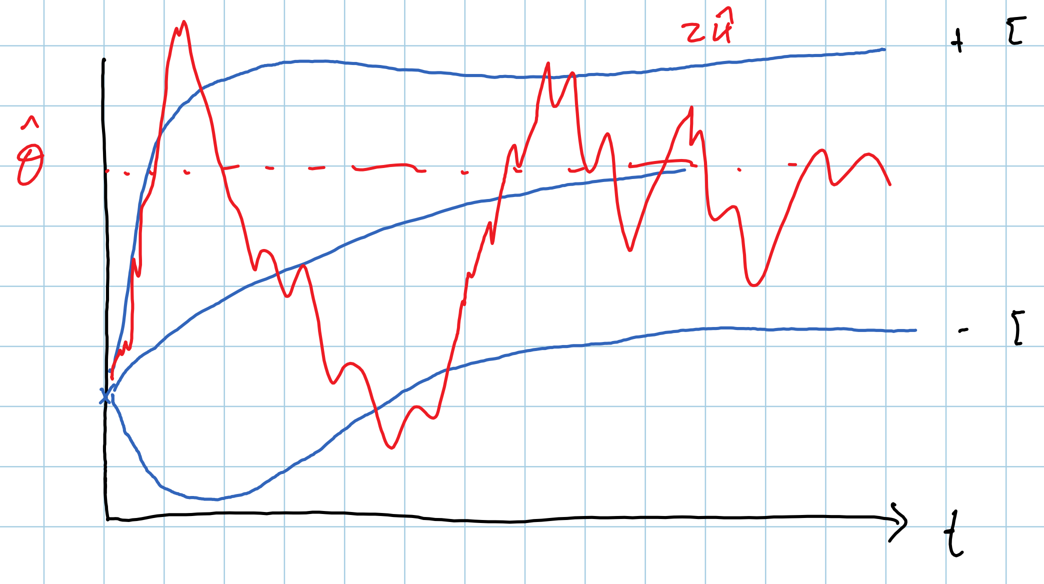
$$\Rightarrow x_t - x_0 = \int_0^t e^{\hat{\kappa} s} \hat{\kappa}\hat{\theta}\,ds + \int_0^t \sigma e^{\hat{\kappa} s}\,d\hat{W}_s$$

$$\Rightarrow \quad x_t - x_0 = \underbrace{\int_0^t e^{\hat{\kappa} s}\, \hat{\kappa}\, \theta \, ds}_{\hat{\theta}\,(e^{\hat{\kappa} t} - 1)} + \underbrace{\int_0^t \sigma\, e^{\hat{\kappa} s}\, dW_s}$$

$x_0 \to r_0$

$$\Rightarrow \quad \boxed{r_t = e^{-\hat{\kappa} t}\, r_0 + \hat{\theta}\,(1 - e^{-\hat{\kappa} t}) + \sigma \int_0^t e^{-\hat{\kappa}(t-s)}\, d\hat{W}_s}$$

$$r_t \underset{\hat{\mathbb{Q}}}{\sim} \mathcal{N}\Big(e^{-\hat{\kappa} t}\, r_0 + \hat{\theta}\,(1 - e^{-\kappa t}) \; ; \; \sigma^2 \underbrace{\int_0^t e^{-2\hat{\kappa}(t-s)}\, ds}_{\frac{1 - e^{-2\hat{\kappa} t}}{2\hat{\kappa}}} \Big)$$



* $P_t(T)$ - bond price @ t for mat T

$$P_t(T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s\, ds} \times 1 \right]$$

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$$\mathbb{E}^{\mathbb{Q}}\left[\;\; \middle| \mathcal{F}_t \right]$$

$$I_t(T) \triangleq \int_t^T r_s\, ds$$

$$\left\{ \; dr_t = \kappa(\theta - r_t)\, dt + \sigma\, dW_t \right.$$

$$r_T - r_t = \int_t^T \kappa(\theta - r_s)\, ds + \sigma \int_t^T dW_s$$

$\tau = T - t$

$$\Rightarrow \int_t^T r_s\, ds = \frac{1}{\kappa}\left\{ \kappa\theta\tau + \sigma\int_t^T dW_s - r_T + r_t \right\}$$

$$= \theta\tau + \frac{\sigma}{\kappa}\int_t^T dW_s + \frac{1}{\kappa} r_t$$

$$- \frac{1}{\kappa}\left(r_t e^{-\kappa\tau} + \theta(1 - e^{-\kappa\tau}) + \sigma\int_t^T e^{-\kappa(T-s)}\, dW_s \right)$$

$$\int_t^T r_s\,ds = \overbrace{r_t\,\frac{1-e^{-\kappa\tau}}{\kappa} + \theta\left[\tau - \frac{1-e^{-\kappa\tau}}{\kappa}\right]}^{\alpha_t(T)}$$

$$+ \frac{\sigma}{\kappa}\int_t^T \left(1 - e^{-\kappa(T-s)}\right) dW_s$$

$$\int_t^T r_s\,ds \sim \mathcal{N}\Big(\alpha_t(T)\,;\ \underbrace{\frac{\sigma^2}{\kappa^2}\int_t^T \left(1-e^{-\kappa(T-s)}\right)^2 ds}_{\Sigma_t^2(T)}\Big)$$

$$P_t(T) = \mathbb{E}_t^{Q}\left[e^{-\int_t^T r_s\,ds}\right]$$

$$= e^{-\alpha_t(T) + \frac{1}{2}\Sigma_t^2(T)}$$

$\hookrightarrow B_t(T)\, r_t + \beta_t(T)$

$$= \exp\left\{A_t(T) - B_t(T)\, r_t\right\}$$

$\hookrightarrow -\beta_t(T) + \frac{1}{2}\Sigma_t^2(T)$

SS1084

Nov 21

tutorial only.

Vasicek Model II: PDE approach

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$\mathbb{Q}$ measure:

$$dr_t = \kappa(\theta - r_t)\,dt + \sigma\,dW_t$$

$$P_t(T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s\,ds} \right] = \underbrace{P(t, r_t; T)}_{\text{PDE}}$$

$$\begin{cases} (\partial_t + \mathcal{L})P = rP, & \mathcal{L} = \kappa(\theta - r)\partial_r + \frac{1}{2}\sigma^2 \partial_{rr} \\ P(T, r) = 1 \end{cases}$$

$$P(t, r) = e^{A_t - B_t r} \longrightarrow B_T = A_T = 0 \text{ b/c } P(T,r) = 1 \ \forall r$$

affine ansatz

(b/c coeff of $\mathcal{L}$ are linear in state variables + S.C. is exp linear.)

$$\partial_t P = (\dot{A} - \dot{B} r) P$$

$$\partial_r P = -BP, \quad \partial_{rr} P = B^2 P$$

$$P\left\{ (\dot{A} - \dot{B} r) + \kappa(\theta - r)(-B) + \frac{1}{2}\sigma^2 B^2 \right\} = rP$$

$$\underbrace{\left(\dot{A} - \kappa\theta B + \frac{1}{2}\sigma^2 B^2\right)}_{=0} - \underbrace{(\dot{B} - \kappa B + 1)}_{=0} r = 0$$

b/c must hold $\forall r$ & t

$$\dot{B} - \kappa B + 1 = 0, \quad B(T) = 0$$

$$\Rightarrow \quad B = \frac{1 - e^{-\kappa(T-t)}}{\kappa}$$

$$\dot{A} - \kappa\theta B + \frac{1}{2}\sigma^2 B^2 = 0$$

$$\underset{0}{A_T} - A_t - \kappa\theta\int_t^T B(u)\,du + \frac{1}{2}\sigma^2\int_t^T B^2(u)\,du = 0$$

$$\Rightarrow \quad A(t) = \frac{1}{2}\sigma^2\int_t^T B^2(u)\,du - \kappa\theta\int_t^T B(u)\,du$$

Bond Option

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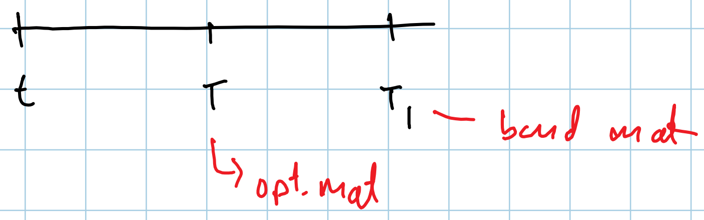
Option on a Bond:



$$V_t^c = \mathbb{E}_t^{Q}\left[e^{-\int_t^T r_s\,ds} \left(P_T(T_1) - K \right)_+ \right]$$

use bond matures T, T_1 ? forward-neutral measure

$$\frac{V_t^c}{P_t(T)} = \mathbb{E}_t^{Q^T}\left[\frac{\left(P_T(T_1) - K\right)_+}{\underbrace{P_T(T)}_{1}} \right]$$

$$= \mathbb{E}_t^{Q^T}\left[\left(P_T(T_1) - K \right)_+ \right]$$

$$\eta_t = \left(\frac{dQ^T}{dQ} \right)_t = \frac{P_t(T) / P_0(T)}{B_t / B_0}$$

$$\frac{d\eta_t}{\eta_t} = \ldots$$ need $dP_t(T) = ?$

recall: $P_t(T) = e^{A_t(T) - B_t(T)\, r_t}$

moreover, $dr_t = \kappa(\theta - r_t)\,dt + \sigma\, dW_t$

$$dP_t(T) = \underbrace{(\partial_t + \mathcal{L})P_t(T)}_{r_t P_t(T)}\,dt + \underbrace{\partial_r P_t(T)}_{-B_t(T)P_t(T)} \cdot \sigma\, dW_t$$

$$\Rightarrow \frac{dP_t(T)}{P_t(T)} = r_t\,dt - \sigma\, B_t(T)\, dW_t$$

$$\hookrightarrow \frac{1 - e^{-\kappa(T-t)}}{\kappa}$$

[note $x_t = \dfrac{P_t(T)}{B_t} = e^{-\int_t^T r_s\,ds} \cdot P_t(T)$ so

$$dx_t = -r_t\, x_t\, dt + e^{-\int_t^T r_s\,ds}\, dP_t(T)$$

$$= -r_t\, x_t\, dt + (r_t\, x_t\, dt - \sigma B_t(T)\, x_t\, dW_t)$$

$$= -\sigma B_t(T)\, x_t\, dW_t\,)$$

$$\Rightarrow \frac{d(P_t(T)/B_t)}{(P_t(T)/B_t)} = -\sigma\, B_t(T)\, dW_t$$

so $\dfrac{d\eta_t}{\eta_t} = -\sigma\, B_t(T)\, dW_t$

η_t

$$\Rightarrow \eta_t = \exp\left\{ -\frac{1}{2}\int_0^t \sigma^2 B_s^2(T)\,ds - \int_0^t \sigma B_s(T)\,dW_s \right\}$$

Girsanov $\Rightarrow$ $dW_t^T = \sigma B_t(T)\,dt + dW_t$

is a $\mathbb{Q}^T$-mtg (B. mtn)

$$\Rightarrow \frac{dP_t(T)}{P_t(T)} = r_t\,dt - \sigma B_t(T)\left(dW_t^T - \sigma B_t(T)\,dt\right)$$

$$= \left(r_t + \sigma^2 B_t^2(T)\right)dt - \sigma B_t(T)\,dW_t^T$$

$\mathbb{E}^{\mathbb{Q}^T}\left[(P_T(T_1) - K)_+\right]$ $\rightarrow$ still hard right now

recall $P_T(T) = 1$.

$$P_T(T_1) = \frac{P_T(T_1)}{P_T(T)} = \lim_{t \uparrow T} \underbrace{\frac{P_t(T_1)}{P_t(T)}}_{X_t}$$

note X_t is a relative price w.r.t. $P_t(T)$ $\Rightarrow$ X_t is a $\mathbb{Q}^T$-mtg!

(also $X_T = P_T(T_1)$)

$$\Rightarrow V_t^C = P_t(T)\,\mathbb{E}^{\mathbb{Q}^T}\left[(X_T - K)_+\right]$$

$$\Rightarrow \quad V_t^c = P_t(T)\, \mathbb{E}^{Q}\left[(X_T - K)_+ \right]$$

compute to find

$$\frac{dX_t}{X_t} = \overbrace{-\sigma\left(B_t(T_1) - B_t(T)\right)}^{\to\ \Sigma_t} dW_t^T$$

(vol T_1) $-$ (vol T)

$$\frac{dX_t}{X_t} = \Sigma_t \, dW_t^T$$

$$\Rightarrow \quad X_T = X_t \, e^{-\frac{1}{2}\int_t^T \Sigma_s^2 \, ds + \int_t^T \Sigma_s \, dW_s^T}$$

$$\sigma^{BS} = \left(\frac{1}{T-t} \int_t^T \Sigma_s^2 \, ds \right)^{1/2}$$

More details...

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some more details on computing dX_t explicitly ...

$$\frac{dP_t(T)}{P_t(T)} = \underbrace{\left(r_t + \sigma^2 B_t^2(T)\right)}_{\gamma_t(T)} dt - \sigma B_t(T)\, dW_t^T$$

and

$$\frac{dP_t(T_1)}{P_t(T_1)} = r_t \, dt - \sigma B_t(T_1)\, dW_t$$

$$= r_t \, dt - \sigma B_t(T_1)\left(-\sigma B_t(T)\, dt + dW_t^T\right)$$

$$= \underbrace{\left(r_t - \sigma^2 B_t(T) B_t(T_1)\right)}_{\gamma_t(T_1)} dt - \sigma B_t(T_1)\, dW_t^T$$

so let $X_t = \dfrac{P_t(T_1)}{P_t(T)} = f(P_t(T_1), P_t(T))$

where $f(a,b) = a/b$,

$\partial_t f = 0$, $\partial_a f = \frac{1}{b}$, $\partial_{aa} f = 0$, $\partial_b f = -\frac{a}{b^2}$, $\partial_{bb} f = +2\frac{a}{b^3}$

$\partial_{ab} f = -\frac{1}{b^2}$

Ito's lemma $\Rightarrow$

$dX_t =$

$$\gamma_t(T_1) P_t(T_1) \perp \quad dt - \overbrace{\sigma B_t(T_1) \cdot P_t(T_1)}^{\text{vol of } P_t(T_1)} \cdot \perp \quad dW_t^T$$

$$\underbrace{\gamma_t(T_1)\, P_t(T_1)}_{\text{drift of } P_t(T_1)} \underbrace{\frac{1}{P_t(T)}}_{\partial_a f(P_t(T_1), P_t(T))} dt - \overbrace{\sigma B_t(T_1) . P_t(T_1)} \cdot \frac{1}{P_t(T)} \, dW_t^T$$

$$+ \left(\underbrace{\gamma_t(T)\, P_t(T)}_{\text{drift of } P_t(T)} \cdot \underbrace{\left(- \frac{P_t(T_1)}{P_t^2(T)} \right)}_{\partial_b f(P_t(T_1), P_t(T))} + \frac{1}{2} \underbrace{\sigma^2 B_t^2(T) . P_t^2(T)}_{(\text{vol of } P_t(T))^2} \cdot \underbrace{\left(2 \frac{P_t(T_1)}{P_t^3(T)} \right)}_{\partial_{bb} f(P_t(T_1), P_t(T))} \right) dt$$

$$- \underbrace{\sigma B_t(T) . P_t(T)}_{\text{vol of } P_t(T)} \cdot \left(- \frac{P_t(T_1)}{P_t^2(T)} \right) dW_t^T$$

$$+ \underbrace{\sigma^2 B_t(T) . P_t(T) . B_t(T_1)\, P_t(T_1)}_{\text{vol of } P_t(T) \text{ \& vol of } P_t(T_1)} \cdot \underbrace{\left(- \frac{1}{P_t^2(T)} \right)}_{\partial_{ab} f(P_t(T_1), P_t(T))} dt$$

$$\Rightarrow \quad dX_t = X_t \underbrace{\left(\gamma(T_1) - \gamma(T) + \sigma^2 B_t^2(T) - \sigma^2 B_t(T) B_t(T_1) \right)}_{3} dt$$

$$- \sigma \left(B_t(T_1) - B_t(T) \right) dW_t^T$$

$$3 = \left(r_t + \sigma^2 B_t(T_1) B_t(T) \right) - \left(r_t + \sigma^2 B_t(T) \right)$$

$$+ \sigma^2 B_t^2(T) - \sigma^2 B_t(T) B_t(T_1) = 0 \; !$$

$$\Rightarrow \quad dX_t = - \sigma \left(B_t(T_1) - B_t(T) \right) dW_t^T .$$

$$\Rightarrow \quad \frac{dX_t}{X_t} = -\sigma\left(B_t(T_1) - B_t(T)\right) dW_t^T .$$