Heston Model

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Heston model:

$$\frac{dF_t}{F_t} = \gamma \sqrt{v_t}\, d\hat{W}_t^F$$

$\cancel{\sigma_t^2 = \gamma \sqrt{v_t}}$

$[\hat{W}^F, \hat{W}^v]_t = \rho t$

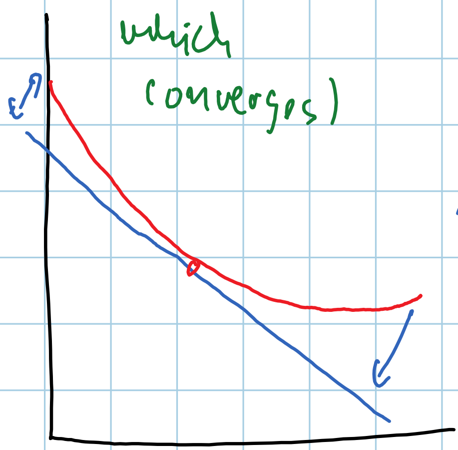
(affecting skewness)

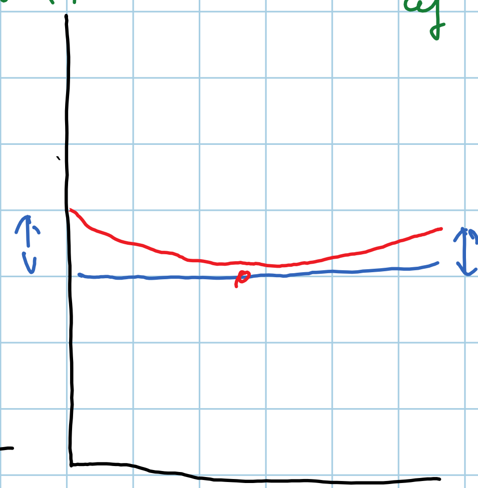
$$dv_t = \kappa(\theta - v_t)\,dt + \alpha \sqrt{v_t}\, d\hat{W}_t^v$$

m.r. rate (affecting rate at which converges)

m.r. level (affect overall vol)

vol-vol (affects sharpness of skews)





VIX

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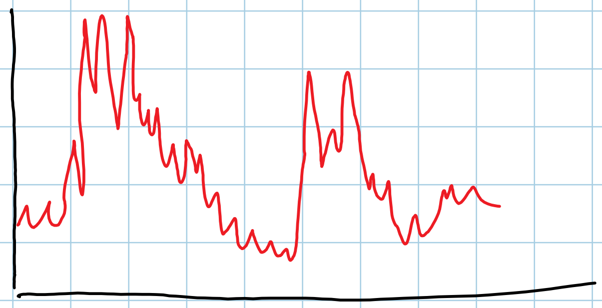
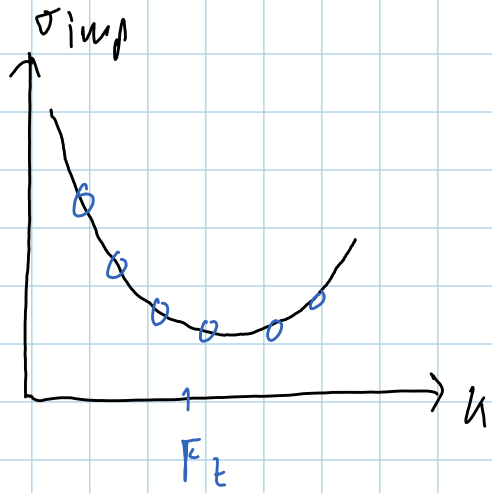
VIX - volatility index





var swap: $\left(\frac{1}{N}\sum_{k=1}^{N}\left(\frac{F_k}{F_{k-1}}-1\right)^2 - K\right)$ @ T

fair strike

$$K = \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{T}\int_0^T \sigma_s^2\, ds\right]$$

will show can write as a collection of put / calls.

$$\frac{dF_t}{F_t} = \sigma_t\, d\hat{W}_t^F$$

; e.g.

- $\sigma_t = \sigma(t, F_t)$
- $d(\sigma_t^2) = \kappa(\bar{\sigma}^2 - \sigma^2)\,dt + \alpha\,\sigma_t^2\, d\hat{W}_t^\sigma$
- $\sigma_t = e^{x_t}$

$$dx_t = \kappa(\theta - x_t)\,dt + \eta\, dW_t^x$$

$$d \ln F_t = -\frac{1}{2}\sigma_t^2\, dt + \sigma_t\, d\hat{W}_t^F$$

$$\ln F_T - \ln F_0 = -\frac{1}{2}\int_0^T \sigma_s^2\, ds + \int_0^T \sigma_s\, d\hat{W}_s^F$$

$$\mathbb{E}^{\mathbb{Q}}\left[\ln\left(\frac{F_T}{F_0}\right)\right] = -\frac{1}{2}\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \sigma_s^2\, ds\right] + 0$$

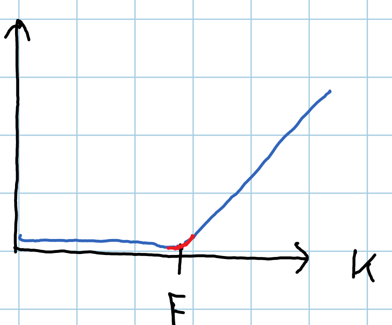
$$\Rightarrow \mathbb{E}^{\mathbb{Q}}\left[\int_0^T \sigma_s^2\, ds\right] = -2\, \mathbb{E}^{\mathbb{Q}}\left[\ln(F_T/F_0)\right]$$

take an arbitrary payoff $g(F)$, $g \in C^2$

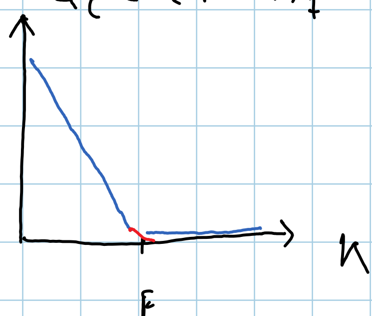
$$g(F) = \int_0^\infty \delta(F-K)\, g(K)\, dK$$

$\mathcal{Q}_p$

$(K-F)_+$

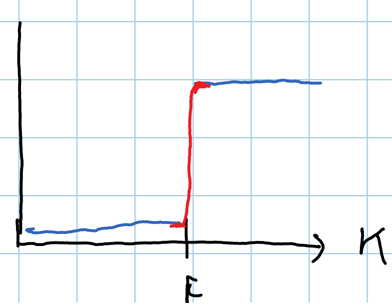


$\mathcal{Q}_c = (F-K)_+$

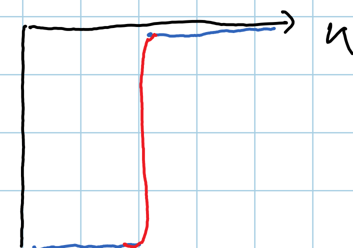


$\partial_K \mathcal{Q}_p$

$= \mathbb{I}_{K>F}$

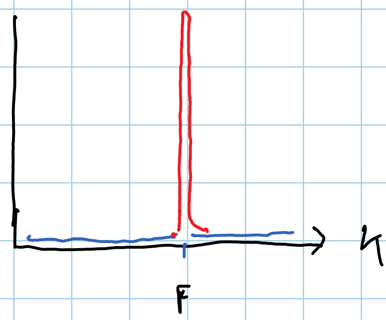


$\partial_K \mathcal{Q}_c$



$\partial_{KK}\Phi_p$

$= \delta(K-F)$



$\partial_{KK}\Phi_c$

$= \delta(K-F)$

$$g(F) = \int_0^{F^*} \partial_{KK}\Phi_p(F,K)\; g(K)\, dK \qquad A \qquad\qquad F^* < F$$

$$+ \int_{F^*}^{\infty} \partial_{KK}\Phi_c(F,K)\; g(K)\, dK \qquad B$$

$$A = -\int_0^{F^*} \partial_K \Phi_p(F,K)\;\; \partial_K g(K)\, dK \; + \; \underbrace{\partial_K \Phi_p(F,K)\, g(K)\Big|_0^{F^*}}_{\to 0}$$

$$B = -\int_{F^*}^{\infty} \partial_K \Phi_c(F,K)\;\; \partial_K g(K)\, dK \; + \; \underbrace{\partial_K \Phi_c(F,K)\, g(K)\Big|_{F^*}^{\infty}}_{\to\, +g(F^*)}$$

$$A = \int_0^{F^*} \Phi_p(F,K)\cdot \partial_K^2 g(K)\, dK \; + \; \underbrace{\Phi_p(F,K)\, \partial_K g(K)\Big|_0^{F^*}}_{\to 0}$$

$$B = \int_{F^*}^{\infty} \Phi_c(F,K)\, \partial_K^2 g(K)\, dK \; - \; \underbrace{\Phi_c(F,K)\, \partial_K g(K)\Big|_{F^*}^{\infty}}$$

$$D = \int_{F^*} Q_c(F,K)\, \partial_K g(K)\, dK = \underbrace{Q_c(F,K)\, \partial_K g(K) \Big|_{F^*}}_{+\,(F - F^*)\, g'(F^*)}$$

$$+\, g(F^*)$$

$$\Rightarrow \quad \boxed{\begin{aligned} g(F) = {} & g(F^*) + (F - F^*)\, g'(F^*) \\ & + \int_0^{F^*} Q_p(F,K)\, g''(K)\, dK \\ & + \int_{F^*}^{\infty} Q_c(F,K)\, g''(K)\, dK \end{aligned}}$$

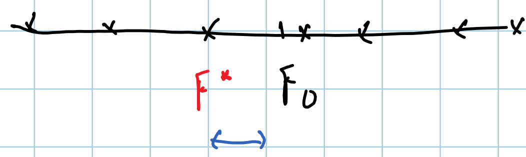
$$\begin{aligned} V^g &= \mathbb{E}^{Q}\left[\, g(F_T) \right] \\ &= g(F^*) + (F_0 - F^*)\, g'(F^*) \\ &\quad + \int_0^{F^*} V^{put}(F_0, K)\, g''(K)\, dK \\ &\quad + \int_{F^*}^{\infty} V^{call}(F_0, K)\, g''(K)\, dK \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad \mathbb{E}^{Q}\left[\ln(F/F_0) \right] &= \ln(F^*/F_0) + \frac{(F_0 - F^*)}{F^*} \\ &\quad + \int_0^{F^*} V^{put}(F_0, K) \left(-\frac{1}{K^2} \right) dK \\ &\quad + \int_{F^*}^{\infty} V^{call}(F_0, K) \left(-\frac{1}{K^2} \right) dK \end{aligned}$$

$$+ \int_{F^*}^{\infty} V^{call}(F_0,K)\left(-\frac{1}{K^2}\right)dK$$

$$\int_0^{F^*} V^{put}(F_0,K)\left(-\frac{1}{K^2}\right)dK \sim \sum_i \frac{V^{put}(F_0,K_i)}{K_i^2}\,\Delta K_i$$

$$F^* = \max_i \{ K_i : K_i < F_0 \}$$



$$-\ln\left(\frac{F^0}{F^*}\right) + \left(\frac{F_0}{F^*} - 1\right)$$

$$\ln(1+x) \sim x - \frac{1}{2}x^2$$

$$\hookrightarrow \ln\left(1 + \underbrace{\left(\frac{F^0}{F^*} - 1\right)}_{\ll 1}\right) \sim \left(\frac{F^0}{F^*} - 1\right) - \frac{1}{2}\left(\frac{F^0}{F^*} - 1\right)^2$$

$$\sim \frac{1}{2}\left(\frac{F_0}{F^*} - 1\right)^2$$

$$\mathbb{E}\left[\ln(F_T/F_0)\right] \sim \frac{1}{2}\left(\frac{F_0}{F^*} - 1\right)^2 - \sum_i \frac{V^{otm}(F,K_i)}{K_i^2}\,\Delta K_i$$

$$\boxed{VIX^2 = \frac{2}{T}\left[\sum_i \frac{V^{otm}(F,K_i)}{K_i^2}\,\Delta K_i - \frac{1}{2}\left(\frac{F_0}{F^*} - 1\right)^2\right]}$$

Tutorial

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$$\mathbb{E}^{Q}\left[g\left(\max_{0\le t\le T} F_t \right) \right] = \mathbb{E}^{Q}\left[\cdots\cdots \int_0^{F^*} \left(K - \max_{0\le t\le T} F_t\right)_+ g''(K)\,dK \right]$$

$$\int_0^{F^*} \mathbb{E}^{Q}\left[\left(K - \max_{0\le t\le T} F_t\right)_+ \right] g''(K)\,dK$$

Tutorial

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$$A = \underline{\varphi(X_T)} \in \mathcal{F}_T$$

$$\rightarrow \quad h_t \triangleq \mathbb{E}[A \mid \mathcal{F}_t] \quad \text{is a } \mathbb{Q}\text{-mtg}$$

$$\mathbb{E}[h_u \mid \mathcal{F}_t] \overset{?}{=} h_t$$

$$\| $$

$$\mathbb{E}[\mathbb{E}[A \mid \mathcal{F}_u] \mid \mathcal{F}_t] = \mathbb{E}[A \mid \mathcal{F}_t]$$

$$= h_t \quad \checkmark$$

$$\lim_{t \uparrow T} h_t = A$$

Tutorial

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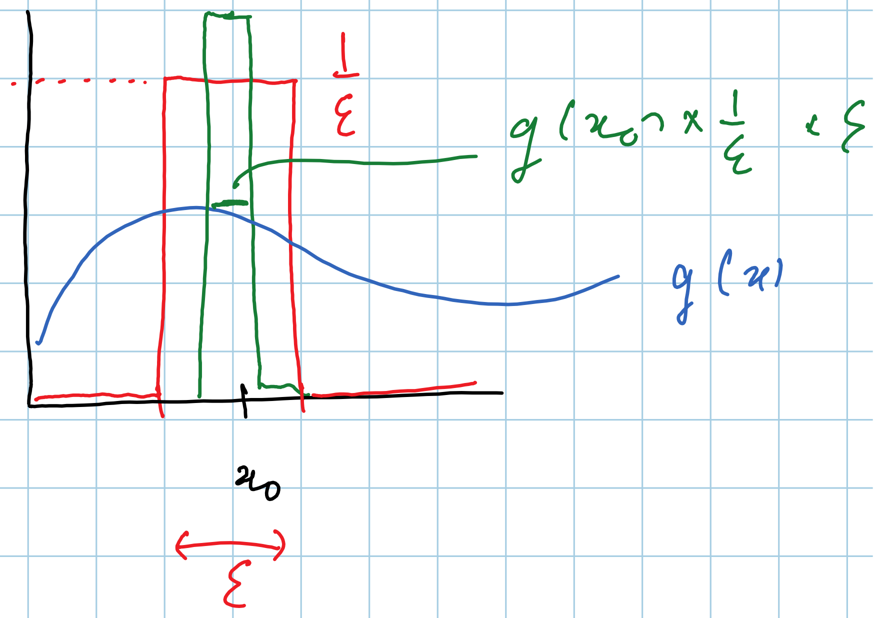
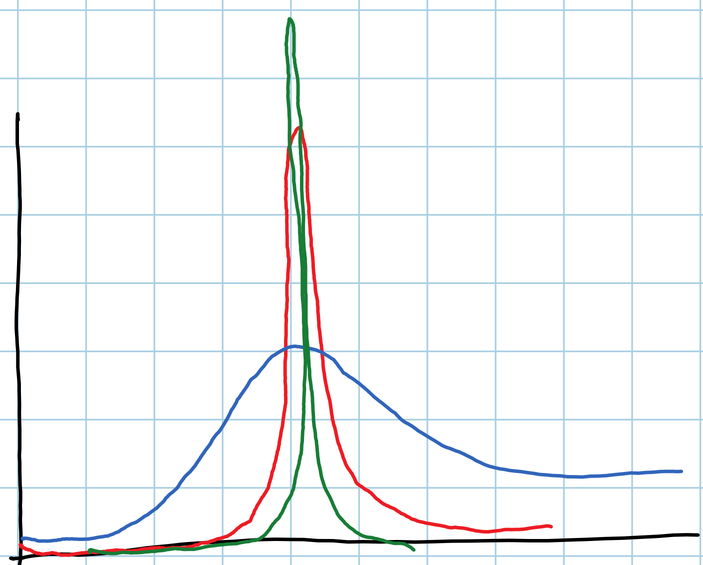
$\delta_\varepsilon(x - x_0)$



$$\int_{-\infty}^{\infty} \underbrace{\delta_\varepsilon(x - x_0) \; g(x)}_{} \, dx \xrightarrow[\varepsilon \downarrow 0]{} g(x_0)$$



Tutorial

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$$\frac{dF_t}{F_t} = \gamma \sqrt{v_t}\, d\hat{W}_t^F$$

$$dv_t = \kappa(\theta - v_t)\,dt + \alpha\sqrt{v_t}\, d\hat{W}_t^v$$

i) $\bar{v}_t = \mathbb{E}[v_t]$

$$d\underbrace{\mathbb{E}[v_t]}_{\bar{v}_t} = \kappa(\theta - \underbrace{\mathbb{E}[v_t]}_{v_t})\,dt + 0$$

$$d\bar{v}_t = \kappa(\theta - \bar{v}_t)\,dt$$

$$\frac{d\bar{v}_t}{\theta - \bar{v}_t} = \kappa\,dt$$

$$\ln\left(\frac{\theta - \bar{v}_t}{\theta - \bar{v}_0}\right) = \kappa t$$

$$\Rightarrow \bar{v}_t = \bar{v}_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t})$$

ii) $$\frac{dF_t}{F_t} = \gamma \sqrt{\bar{v}_t}\, d\hat{W}_t^F$$

$$\mathbb{E}^Q\left[\mathbb{1}_{F_T < K}\right] = Q(F_T < K)$$

$$= Q(\ln F_T < \ln K)$$

$$d \ln F_t = -\frac{1}{2}\gamma^2 \bar{v}_t\, dt + \gamma \sqrt{\bar{v}_t}\, d\hat{W}_t^F$$

$\rightarrow$ $$\ln(F_T/F_0) = -\frac{1}{2}\gamma^2 \int_0^T \bar{v}_s\, ds + \gamma \int_0^T \sqrt{\bar{v}_s}\, d\hat{W}_s^F$$

$$\underset{Q}{\sim} N\left(-\frac{1}{2}\gamma^2 \int_0^T \bar{v}_s\, ds \,;\, \Sigma^2\right)$$

$$\Sigma^2 = \mathbb{E}^Q\left[\left(\gamma \int_0^T \sqrt{\bar{v}_s}\, d\hat{W}_s^F\right)^2\right]$$

$$= \gamma^2 \int_0^T \bar{v}_s\, ds$$

$$\ln(F_T/F_0) \overset{d}{=} -\frac{1}{2}\Sigma^2 + \Sigma Z, \quad Z \underset{Q}{\sim} N(0,1)$$

$$\ln(F_T/F_0) = -\tfrac{1}{2}\Sigma^2 + \Sigma Z, \quad Z \underset{Q}{\sim} N(0,1)$$

$$\therefore \quad V_0 = Q\left(-\tfrac{1}{2}\Sigma^2 + \Sigma Z < \ln(K/F_0)\right)$$

$$= \Phi\left(\frac{\ln(K/F_0) + \frac{1}{2}\Sigma^2}{\Sigma}\right)$$

$$F_t(T) = \mathbb{E}_t^{Q}[S_T]$$

Tutorial

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X is a r.v. estimate $\mathbb{E}[X]$

$$\hat{\mu}^X = \widehat{\mathbb{E}[X]} = \frac{1}{N}\sum_{n=1}^{N} X^{(n)} \quad , \quad X^{(1)}, \ldots \text{ iid draws from } X$$

suppose Y is another r.v. (correlated with X)

and $\mu^Y = \mathbb{E}[Y]$ is known.

$$\hat{\mu}^X_* = \hat{\mu}^X + \gamma\,(\hat{\mu}^Y - \underbrace{\mathbb{E}[Y]}_{\mu^Y})$$

$\hat{\mu}^X$, $\hat{\mu}^Y$ → estimated simultaneously from joint dist (X, Y)

$$\min_{\gamma} \mathbb{V}\Big[\underbrace{\big(X + \gamma\,(Y - \mu^Y)\big)}_{\alpha}\Big]$$

$$\mathbb{E}[\alpha] = \mathbb{E}[X]$$

$$\mathbb{V}[\;\cdot\;] = \mathbb{E}[(X + \gamma(Y - \mu^Y))^2] - (\mathbb{E}[X])^2$$

$$\partial_\gamma(\quad) = 0$$

$$\mathbb{E}[(Y-\mu^Y)(X + \gamma(Y - \mu^Y))] = 0$$

0, a. b.

$$\mathbb{E}[YX] + \underbrace{\gamma\,\mathbb{E}[Y^2] - \gamma\mu^Y \underbrace{\mathbb{E}[Y]}_{\mu^Y}}_{\gamma\,\mathbb{V}[Y]} = 0$$

$$\Rightarrow \quad \gamma = -\frac{\mathbb{E}[YX]}{\mathbb{V}[Y]}, \quad -\frac{\mathbb{C}[X,Y]}{\mathbb{V}[Y]}$$

$$\frac{\widehat{\mathbb{C}[X,Y]}}{\widehat{\mathbb{V}[Y]}}$$