Implied Volatility

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$$\frac{dS_t}{S_t} = r\,dt + \sigma\,dW_t$$

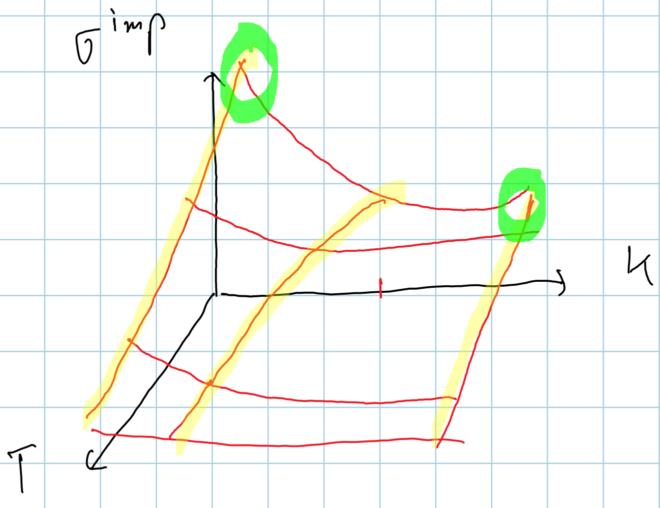
$$\hookrightarrow \boxed{C_t = S_t\,\Phi(d_t^+) - K e^{-r(T-t)}\,\Phi(d_t^-)}$$

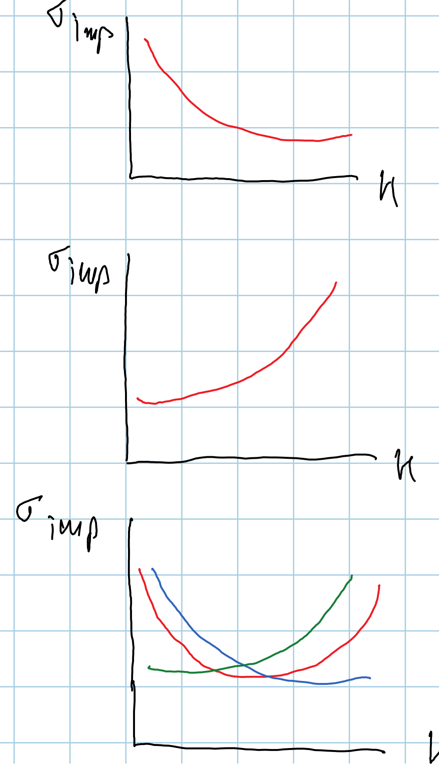
$$d_t^{\pm} = \frac{\ln(S_t/K) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$C_t = C(t, S_t;\ \sigma,\ \underbrace{K, T})$$

$$C_0(\sigma;\ K, T) = C_0^*(K, T)$$

$\hookrightarrow \sigma^{imp}(K,T)$ implied volatility.

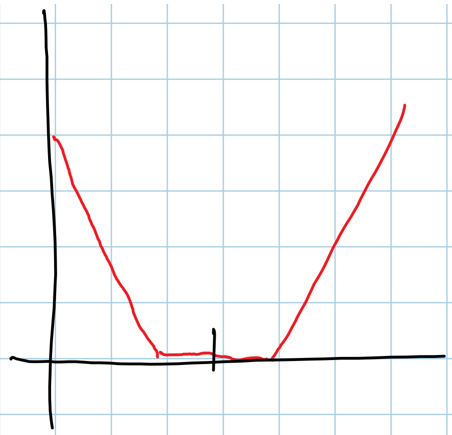


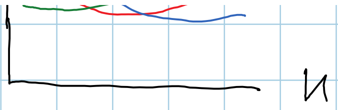


equity

commodity

FX

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Local Vol / Heston

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local volatility model

$$\frac{dS_t}{S_t} = r\,dt + \underbrace{\sigma(t, S_t)}_{\text{e.g. } S_t^{\alpha}}\, dW_t$$

constant elasticity variance
CEV models

$$\begin{cases} \partial_t f + r S \partial_S f + \frac{1}{2}\sigma^2(t,S) S^2 \partial_{SS} f = r f \\ f(T,S) = \varphi(S) \end{cases}$$

$\exists\ \sigma_{loc}(t,S)$ s.t. $\sigma_{imp}(T,K) = \sigma^*_{imp}(T,K)$

Dupire formula.

Heston model $\qquad F_t(T) = \mathbb{E}_t^{\mathbb{Q}}[S_T] = e^{r(T-t)} S_t$

$$\frac{dF_t}{F_t} = \gamma \sqrt{v_t}\, dW_t^F$$

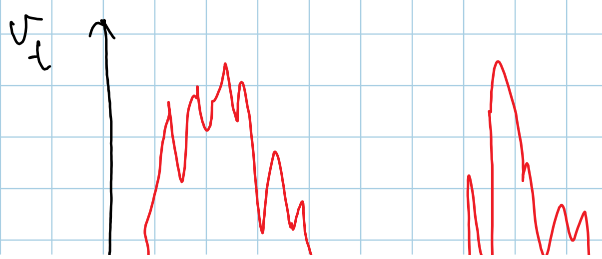
$$dv_t = \kappa(\theta - v_t)\,dt + \alpha\sqrt{v_t}\, dW_t^v .$$

m.r. rate $\qquad$ m.r. level $\qquad$ vol-vol

¿ $\frac{\alpha}{2\kappa} < \theta$?

Feller condition.

$$[W^F, W^v]_t = \rho t$$



typically one assumes: $\lambda_t^v = c\sqrt{v_t} + d/\sqrt{v_t}$

$$dW_t^v = \lambda_t^v \, dt + dW_t^{\mathbb{P},v}$$

Solving Heston PDE

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$$\frac{dF_t}{F_t} = \gamma \sqrt{v_t}\, dW_t^F$$

$$dv_t = \kappa(\theta - v_t)\,dt + \alpha\sqrt{v_t}\, dW_t^v .$$

$$\begin{cases} (\partial_t + \mathcal{L})g = rg \\ g(T,F) = \varphi(F) \end{cases} \qquad g = e^{-r(T-t)} \underbrace{\mathbb{E}_t^{\mathbb{Q}}[\varphi(F_T)]}_{h(t,F_t,v_t)}$$

S_T

h_t is a $\mathbb{Q}$-mtg

$$dh_t = \overbrace{(\partial_t + \mathcal{L})h}^{=0}\,dt + \gamma\sqrt{v_t}\,F_t\,\partial_F h\, dW_t^F + \alpha\sqrt{v_t}\,\partial_v h\, dW_t^v$$

where,

$$\mathcal{L}h = \tfrac{1}{2}(\gamma\sqrt{v}\,F)^2\,\partial_{FF}h + \kappa(\theta - v)\,\partial_v h + \tfrac{1}{2}(\alpha\sqrt{v})^2\,\partial_{vv}h + \rho\,\gamma\sqrt{v}\,F\cdot\alpha\sqrt{v}\,\partial_{Fv}h$$

$$= \tfrac{1}{2}\gamma^2 v F^2\,\partial_{FF}h + \kappa(\theta - v)\,\partial_v h + \tfrac{1}{2}\alpha^2 v\,\partial_{vv}h + \rho\gamma\alpha v\, F\,\partial_{Fv}h$$

$x_t = \ln F_t \qquad \left(\frac{dF_t}{F_t} = \gamma \sqrt{v_t}\, dW_t^F \right)$

$\Rightarrow \quad dx_t = -\frac{1}{2}\gamma^2 v_t\, dt + \gamma \sqrt{v_t}\, dW_t^F$

$$\mathcal{L}^{x,v} h = -\frac{1}{2}\gamma^2 v\, \partial_x h + \frac{1}{2}\gamma^2 v\, \partial_{xx} h$$
$$+ \kappa(\theta - v)\, \partial_v h + \frac{1}{2}\alpha^2 v\, \partial_{vv} h$$
$$+ \rho\gamma\alpha\, v\, \partial_{xv} h$$

$$(\partial_t + \mathcal{L}^{x,v}) h = 0$$

$$\varphi(x) = \sum a_n\, \ell_n(x) = \int_{-\infty}^{\infty} e^{-i\omega x}\, a(\omega)\, d\omega$$

so apply B.C $h(T, x) = e^{-i\omega x}$

turns out that $h(t, x, v) = e^{A_t + B_t x + C_t v}$

b/c $\mathcal{L}^{x,v}$ has linear coeff & $h(T)$ is exponential affine.

$\partial_x h = B_t h, \quad \partial_{xx} h = B_t^2 h$

$\partial_v h = C_t h, \quad \partial_{vv} h = C_t^2 h, \quad \partial_{xv} = B_t C_t h$

$\partial_t h = (\dot{A}_t + \dot{B}_t x + \dot{C}_t v) h$

$h \{ \; (\; x \quad)$

$$+ (\quad \mathcal{H}_x \quad) \, x$$

$$+ (\quad \mathcal{H}_v \quad) \, v \, \} = 0$$

must hold $\forall \, x, v$

$\Rightarrow \quad \mathcal{H} = \mathcal{H}_x = \mathcal{H}_v = 0$ these are coupled Riccati equations and can be solved explicitly.

can use these prices to value any claim.

$$Q(x) = \int_{-\infty}^{\infty} e^{-i\omega x} \, a(\omega) \, d\omega$$

$$h(t, x, v) = \int_{-\infty}^{\infty} e^{A_t + B_t x + C_t v} \, a(\omega) \, d\omega$$

Sample Surfaces

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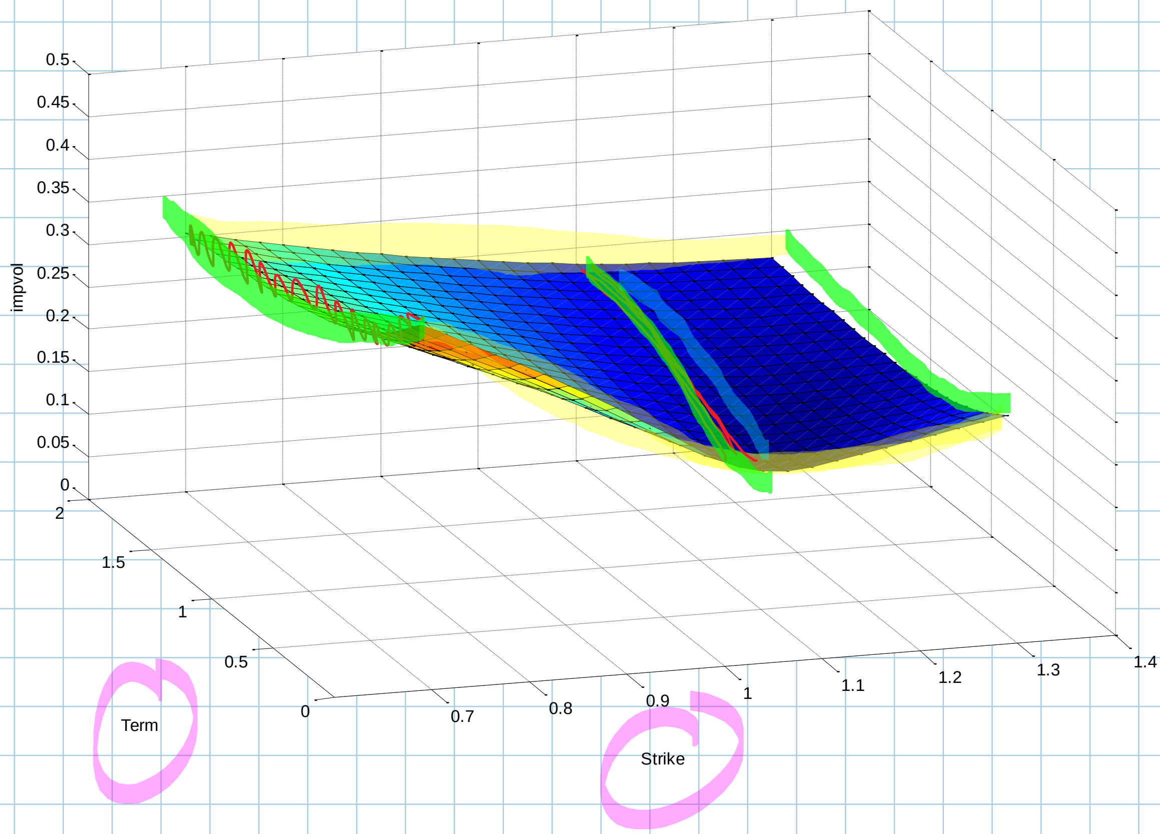

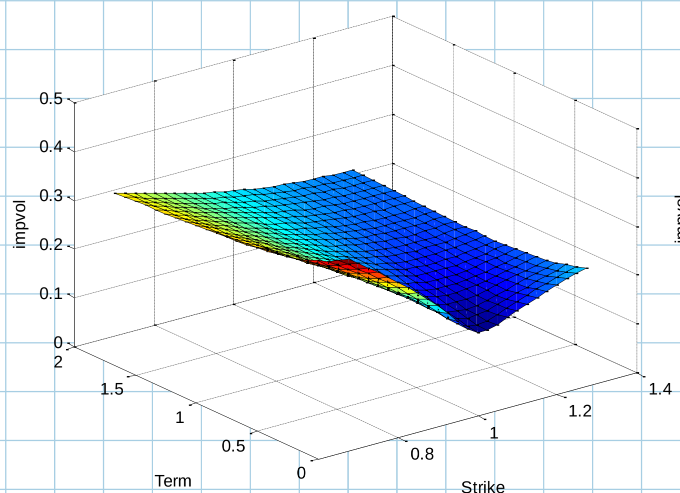

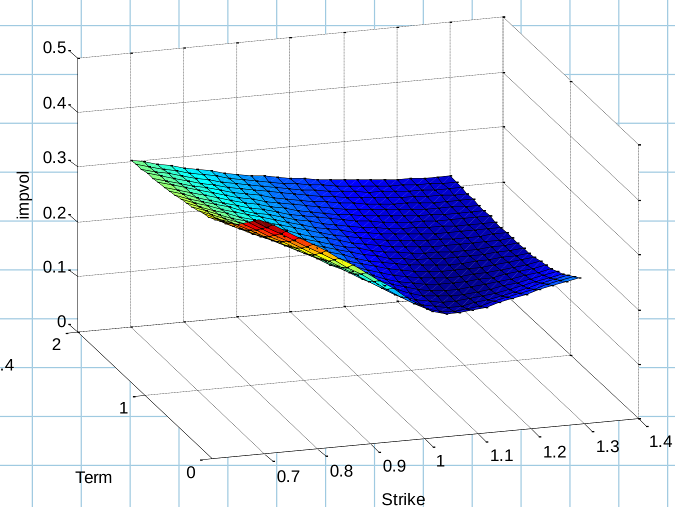

Simulation in Heston

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Simulating Heston.

$$\frac{dF_t}{F_t} = \gamma \sqrt{v_t}\, dW_t^F \qquad dx_t = -\frac{1}{2}\gamma^2 v_t\, dt + \gamma\sqrt{v_t}\, dW_t^F$$

$$dv_t = \kappa(\theta - v_t)\,dt + \alpha\sqrt{v_t}\, dW_t^v$$

Euler:

$$x_n = x_{n-1} - \frac{1}{2}\gamma^2 v_{n-1}^+ \Delta t + \gamma\sqrt{v_{n-1}^+}\sqrt{\Delta t}\, Z_n$$

$$v_n = v_{n-1} + \kappa(\theta - v_{n-1}^+)\Delta t + \alpha\sqrt{v_{n-1}^+}\left(\rho Z_n + \sqrt{1-\rho^2}\, Z_n^\perp\right)$$

$$v^+ = \max(v, 0)$$

puts calls

$\leftarrow K$ S_0 $K \to$

---

control variate:

$$dv_t = \kappa(\theta - v_t)\,dt + \alpha\sqrt{v_t}\, dW_t^v$$

$$d\,\mathbb{E}[v_t] = \kappa(\theta - \mathbb{E}[v_t])\,dt + \alpha\,\mathbb{E}[\sqrt{v_t}\, dW_t^v]$$

$$d\,\mathbb{E}[v_t] = \kappa(\theta - \mathbb{E}[v_t])\,dt + \alpha\,\underbrace{\mathbb{E}[\sqrt{v_t}\,dW_t^v]}_{\to 0}$$

$$d\bar{v}_t = \kappa(\theta - \bar{v}_t)\,dt$$

$$\frac{dF_t}{F_t} = \gamma\sqrt{\bar{v}_t}\,dW_t^F$$

$$dx_t = -\frac{1}{2}\gamma^2\bar{v}_t\,dt + \gamma\sqrt{\bar{v}_t}\,dW_t^F$$

$$\Rightarrow \quad x_T - x_0 = -\frac{1}{2}\gamma^2\int_0^T \bar{v}_u\,du + \underbrace{\gamma\int_0^T \sqrt{\bar{v}_u}\,dW_u^F}_{Y}$$

Y is normally dist.

$$Y \overset{d}{=} \Sigma Z, \quad Z \sim N(0,1)$$

$$\Sigma^2 = \mathbb{E}\left[\left(\gamma\int_0^T \sqrt{\bar{v}_u}\,dW_u^F\right)^2\right]$$

$$= \gamma^2\int_0^T \bar{v}_u\,du$$

Variance Swap

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Variance Swaps:

realized variance over $[0, T]$

$$V^{real} = \frac{1}{N} \sum_{n}^{N} \left(\frac{F_n}{F_{n-1}} - 1 \right)^2$$

Var swap pays $(V^{real} - K)$ @ T.

value of K s.t. Varswap has zero value is the fair variance swap strike.

$$\sum_n \left(\frac{F_n}{F_{n-1}} - 1 \right)^2 \sim \sum_n \left(\ln(F_n / F_{n-1}) \right)^2$$

$$= \sum_n \left(\ln F_n - \ln F_{n-1} \right)^2$$

$$= \sum_n (\Delta x_n)^2$$

$\hookrightarrow \sim \gamma^2 v_{n-1} \Delta t \, z_n^2$

$$\rightarrow \int_0^T \gamma^2 v_u \, du$$

$$\mathbb{E}^{Q}[v^{\text{real}}] \sim \mathbb{E}^{Q}\left[\int_0^T \gamma^2 v_u \, du\right]$$

$$= \int_0^T \gamma^2 \, \mathbb{E}^{Q}[v_u] \, du$$

$$= \dots$$

$$\left.\begin{array}{l} (V^{\text{realised}} - K)_+ \\ (K - V^{\text{real}})_+ \end{array}\right\} \longrightarrow \text{prices}$$

assume $v^{\text{realized}} \overset{d}{=} v_0 e^{X}$

$$X \sim \mathcal{N}\left(-\tfrac{1}{2}\eta^2 T ; \ \eta^2 T\right)$$

$\longrightarrow$ formula

$\eta_{imp}(K, T)$ implied vol-vol