Black-Scholes Model

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$\mathbb{P}$- B.mtn.

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t \qquad , \qquad r = \text{const.}$$

h_t pays $\varphi(S_T)$ @ T satisfies: $h_t = h(t, S_t)$

$$\begin{cases} \partial_t h + r S \partial_S h + \frac{1}{2}\sigma^2 S^2 \partial_{SS} h = r h \\ h(T, S) = \varphi(S) \end{cases}$$

¿ if $\varphi(S) = S$ $\quad h = ?$ $\qquad h(t, S) = S$

i.e. $h_t = S_t$.

$$\left.\begin{array}{l} \text{lhs} = 0 + r S \cdot 1 + \frac{1}{2}\sigma^2 S^2 \cdot 0 = r S \\ \text{rhs} = r \cdot S \qquad \& \quad h(T,S) = S \end{array}\right\} \text{ so satisfies PDE}$$

$$h(t, S) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} S_T \mid S_t = S \right]$$

$$\frac{dS_t}{S_t} = r\, dt + \sigma\, d\hat{W}_t \qquad , \qquad S_t = S$$

$\mathbb{Q}$- B.mtn.

$$\hookrightarrow \quad S_t = S\, e^{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma(\hat{W}_T - \hat{W}_t)}$$

$$X_t = \ln S_t = f(S_t) \quad , \quad f(S) = \ln S$$

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$$dX_t = \left[\partial_t f + rS\,\partial_S f + \tfrac{1}{2}\sigma^2 S^2\,\partial_{SS} f \right] dt + \partial_S f \cdot \sigma S\, d\hat{W}_t$$

$$= \left[0 + rS \cdot \tfrac{1}{S} + \tfrac{1}{2}\sigma^2 S^2 \left(-\tfrac{1}{S^2}\right) \right] dt + \tfrac{1}{S} \cdot \sigma S\, d\hat{W}_t$$

$$= \underline{(r - \tfrac{1}{2}\sigma^2)\,dt + \sigma\, d\hat{W}_t}$$

$$\Rightarrow \quad X_T - X_t = (r - \tfrac{1}{2}\sigma^2)(T-t) + \sigma(\hat{W}_T - \hat{W}_t)$$

$$\Rightarrow \quad S_T = S_t\, e^{(r-\frac{1}{2}\sigma^2)(T-t) + \sigma(\hat{W}_T - \hat{W}_t)}$$

$$\left(S_T \overset{d}{=} S_t\, e^{(r-\frac{1}{2}\sigma^2)(T-t) + \sigma\sqrt{T-t}\, Z} \quad , \quad Z \underset{Q}{\sim} N(0,1) \right)$$

CRR $N \to \infty$

$$h(t,S) = \mathbb{E}^{Q}\left[e^{-r(T-t)} \cdot S_T \right] \qquad \underbrace{}_{} \sim_Q N(0, T-t)$$

$$= \mathbb{E}^{Q}\left[S\, e^{-\frac{1}{2}\sigma^2(T-t) + \sigma(\hat{W}_T - \hat{W}_t)} \right]$$

$$= S\, e^{-\frac{1}{2}\sigma^2(T-t)}\; e^{\frac{1}{2}\sigma^2(T-t)} = S.$$

¿ $\Phi(S) = S^2 \quad h = ?$

is $h = s^2$?

$$\partial_t h + rS\,\partial_S h + \tfrac{1}{2}\sigma^2 S^2\,\partial_{SS} h = rh, \qquad h(T,S) = S^2$$

$$\underbrace{0 + rS\cdot 2S + \tfrac{1}{2}\sigma^2 S^2 \cdot 2}_{(2r+\sigma^2)\,S^2} \qquad rS^2$$

$$h = S^2 e^{-ct} \cdot e^{cT}$$

$$\underbrace{-cS^2 e^{-ct} + rS\cdot 2S\, e^{-ct} + \tfrac{1}{2}\sigma^2 S^2 \cdot 2\, e^{-ct}}_{(2r+\sigma^2-c)\,S^2 e^{-ct}} \qquad r\, e^{-ct} S^2$$

$$c = r + \sigma^2$$

$$h(t,S) = S^2 e^{+(r+\sigma^2)(T-t)}.$$

or

$$h(t,S) = \mathbb{E}^{Q}\left[e^{-r(T-t)} S_T^2 \mid S_t = S\right]$$

$$= \mathbb{E}^{Q}\left[e^{-r(T-t)} S^2 e^{2(r-\frac{1}{2}\sigma^2)(T-t) + 2\sigma(\hat{W}_T - \hat{W}_t)} \mid S_t = S\right]$$

$$= S^2 e^{(r-\sigma^2)\tau}\, \mathbb{E}_t^{Q}\left[e^{2\sigma(\hat{W}_T - \hat{W}_t)}\right]$$

$$= S^2 e^{(r-\sigma^2)\tau} \cdot e^{\frac{1}{2}4\sigma^2(T-t)}$$

$$= S^2 e^{(r+\sigma^2)(T-t)}$$

try $\varphi(S) = S^n$.

. try $\varphi(S) = S^n$.

$$\partial_t h + r S \partial_S h + \tfrac{1}{2}\sigma^2 S^2 \partial_{SS} h = r h$$

can be reduced to heat equation.

$h(t,S) = g(t, \ln S)$, find a new PDE for $g(t,x)$

$$\begin{cases} \partial_t g + (r - \tfrac{1}{2}\sigma^2)\partial_x g + \tfrac{1}{2}\sigma^2 \partial_{xx} g = r g \\ g(T,x) = \varphi(e^x) \end{cases}$$

$$g(t,x) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} \varphi(e^{X_T}) \mid X_t = x \right]$$

$$dX_t = (r - \tfrac{1}{2}\sigma^2)\,dt + \sigma\, d\hat{W}_t \quad (= \sigma\, d\tilde{W}_t)$$

$$Y_t = X_t + (r - \tfrac{1}{2}\sigma^2)(T-t)$$

$$dY_t = dX_t - (r - \tfrac{1}{2}\sigma^2)\,dt = \sigma\, d\hat{W}_t$$

$$g(t,x) = l(t, \underbrace{x + (r - \tfrac{1}{2}\sigma^2)(T-t)}_{y})$$

$$= \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} \varphi(e^{Y_T}) \mid Y_t = y \right]$$

$$\partial_t l + 0 \cdot \partial_y l + \tfrac{1}{2}\sigma^2 \partial_{yy} l = r l$$

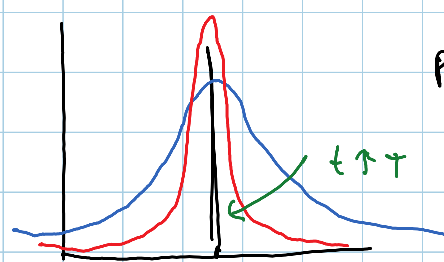
$$\partial_t \ell + 0 \cdot \partial_y \ell + \tfrac{1}{2}\sigma^2 \partial_{yy} \ell = r\ell$$

$$\partial_t \ell + \tfrac{1}{2}\sigma^2 \partial_{yy} \ell = r\ell$$

$\gtrless 0$

$$\ell(t,y) = e^{+r(T-t)}\, p(t,y)$$

$$\boxed{\partial_t p + \tfrac{1}{2}\sigma^2 \partial_{yy} p = 0}$$



$$p(t,y) = \frac{e^{-\frac{1}{2}\frac{(y-y_0)^2}{\sigma^2 (T-t)}}}{\sqrt{2\pi\sigma^2 (T-t)}}$$

¿ $Q(S) = D_{S_0}(S)$?

Time and move-based hedging

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$$\alpha_t = \frac{g_t \, \sigma_t^g}{F_t \, \sigma_t^F}$$

for B-S $F_t = S_t$

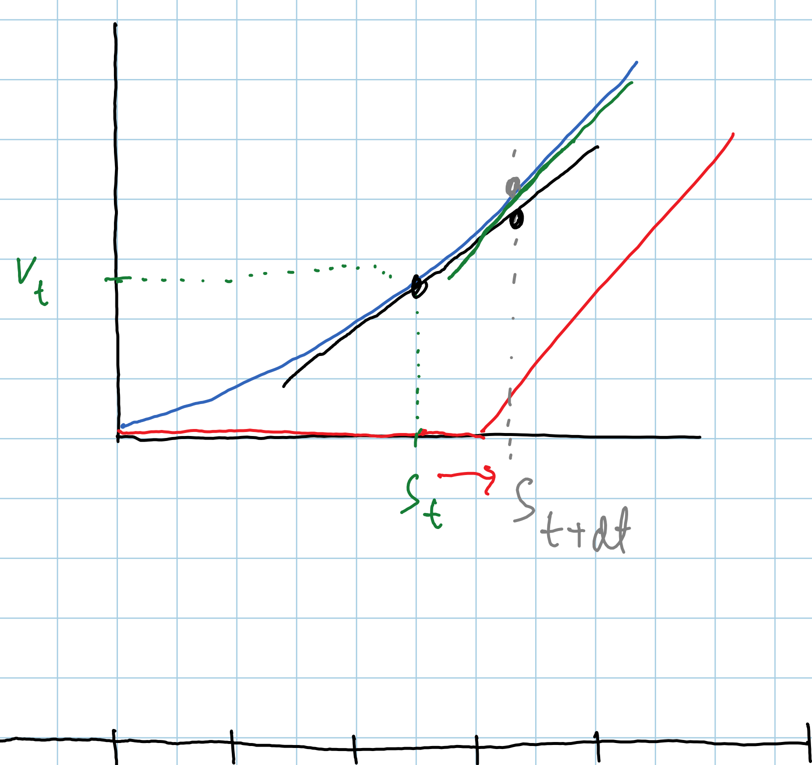
and so $\sigma_t^F = \sigma$.

& $\sigma_t^g = \dfrac{\sigma S_t \, \partial_S g_t}{g_t}$

(recall $\dfrac{dg_t}{g_t} = \mu_t^g \, dt + \sigma_t^g \, dW_t$

$$\frac{dg_t}{g_t} = \frac{(\partial_t + \mathcal{L}) g}{g_t} dt + \frac{\sigma S_t \cdot \partial_S g}{g_t} dW_t$$

$\Rightarrow \quad \alpha_t = \partial_S g_t$ — Delta of the option.
(Δ)



0 ΔT $2\Delta T$ . . $T = n\Delta T$

sold an option get V_0

<u>t = 0</u>

purchase Δ_0 units of S (costs $S_0 \Delta_0$) $+ \beta |\Delta_0 S_0|$

amt. left is $(V_0 - \Delta_0 S)$ put into MM.
$= M_0$

<u>t = Δt</u>

$M_0 \mapsto M_0 e^{r\Delta t}$

$S_0 \mapsto S_1$ value of holdings is now

$\Delta_0 S_1$

rebalance and buy $(\Delta_1 - \Delta_0)$ units of S

costs $(\Delta_1 - \Delta_0) S_1 + \beta |\Delta_1 - \Delta_0| S_1$

now have $M_1 = M_0 e^{r\Delta t} - ((\Delta_1 - \Delta_0) S_1 + |\Delta_1 - \Delta_0| \beta S_1)$ in bank

& hold Δ_1 units of asset

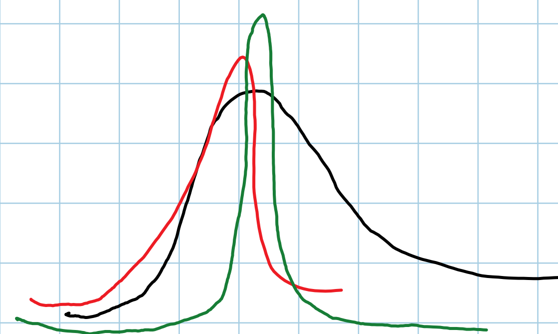
in general ...

@ $k\Delta t$ $M_k = M_{k-1} e^{r\Delta t} - ((\Delta_k - \Delta_{k-1}) S_k + |\Delta_k - \Delta_{k-1}| \beta S_k)$ in bank

& Δ_k units of S.

but @ $T = n\Delta t$; $V_n = M_{n-1} e^{r\Delta t} + \Delta_{n-1} S_n$

$$PnL = V_n - \varphi(S_n).$$

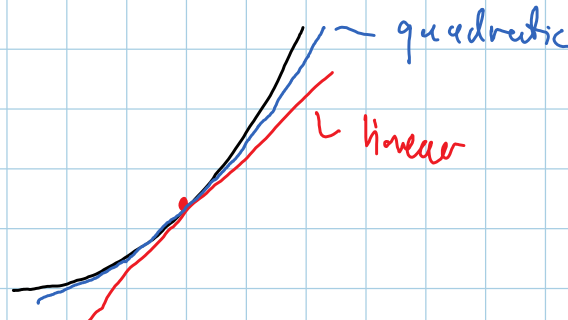


Taylor expansion of $g(t,S)$ around the current asset price (S_t)

$$g(t,S) = g(t,S_t) + (S-S_t)\,\partial_S g(t,S_t) \quad \Delta_t$$

$$+ \frac{1}{2}(S-S_t)^2\,\partial_{SS} g(t,S_t) \quad \Gamma_t \text{ gamma}$$

$$+ \cdots$$



can move along the slope curve by holding

× Bank × S × another option (H)

α β γ

$$\Delta^V_t = \partial_s(\alpha + \beta S + \gamma h(t,S))$$

$$= \beta + \gamma\, \partial_s h_t = \beta + \gamma\, \Delta^h_t = \Delta^g_t$$

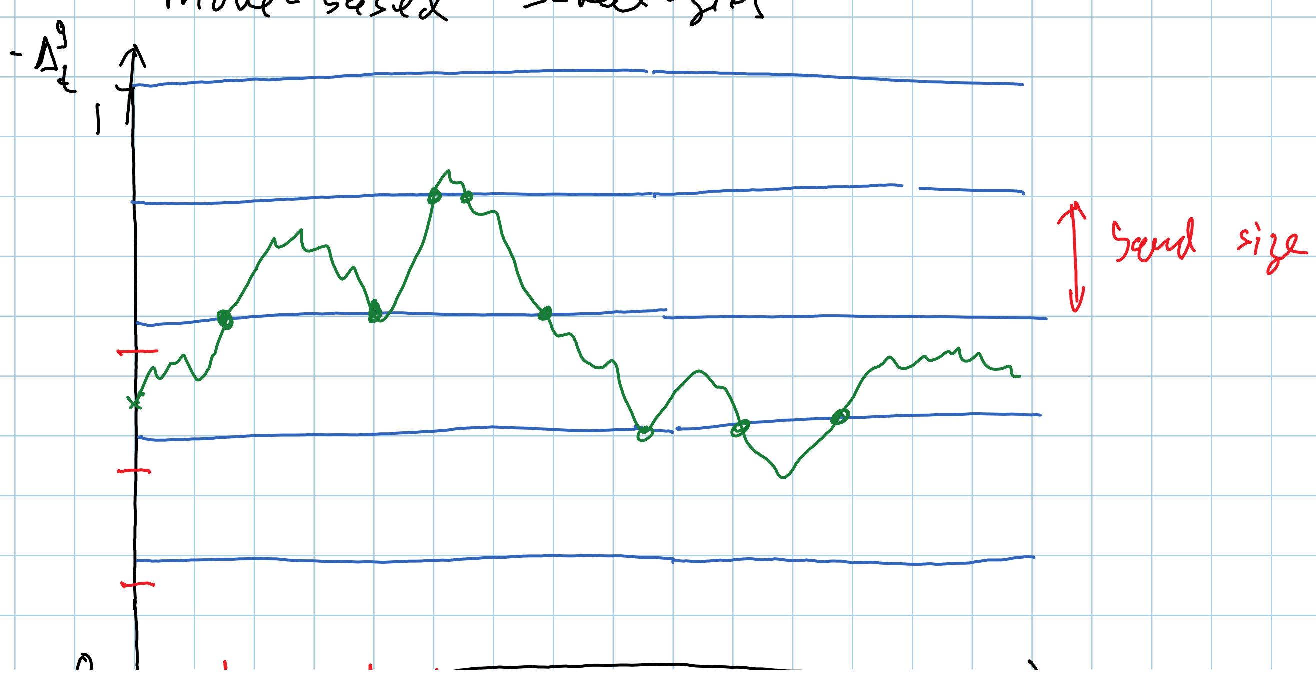
$$\Gamma^V_t = \gamma\, \partial_{ss} h_t = \gamma\, \Gamma^h_t = \Gamma^g_t$$

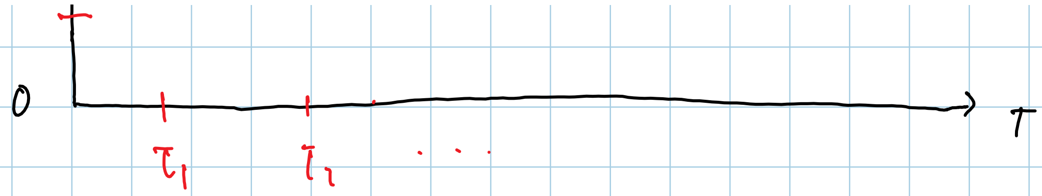
$$\boxed{\gamma_t = \frac{\Gamma^g_t}{\Gamma^h_t}, \quad \beta_t = \Delta^g_t - \frac{\Gamma^g_t}{\Gamma^h_t}\Delta^h_t}$$

$$\gamma_{k-1}\, h(t_{k-1}, S_{k-1}; T)$$

$$\hookrightarrow \gamma_{k-1}\, h(t_k, S_k; T)$$

Move-based strategies





BS Delta and Gamma

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$$C = S\,\Phi(d_+) - K e^{-rT}\,\Phi(d_-)$$

$$d_\pm = \frac{\ln(S/K) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sqrt{\sigma^2 (T-t)}}$$

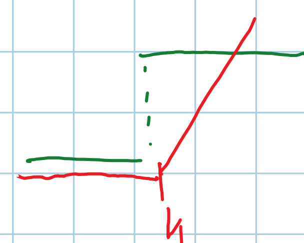
$$\Delta = \partial_S C = \partial_S\, \mathbb{E}^{\mathbb{Q}}\left[(S_T - K)_+ \mid S_t = S\right] e^{-r\tau}$$

$$= \partial_S\, \mathbb{E}^{\mathbb{Q}}\left[(S e^{X} - K)_+ \right] e^{-r\tau}$$

$$X \underset{\mathbb{Q}}{\sim} N\left((r - \tfrac{1}{2}\sigma^2)\tau \,;\, \sigma^2 \tau \right)$$

$$= \mathbb{E}^{\mathbb{Q}}\left[\partial_S (S e^{X} - K)_+ \right] e^{-r\tau}$$

$$= \mathbb{E}^{\mathbb{Q}}\left[S e^{X}\, \mathbb{1}_{S e^{X} > K} \right] \frac{e^{-r\tau}}{S}$$



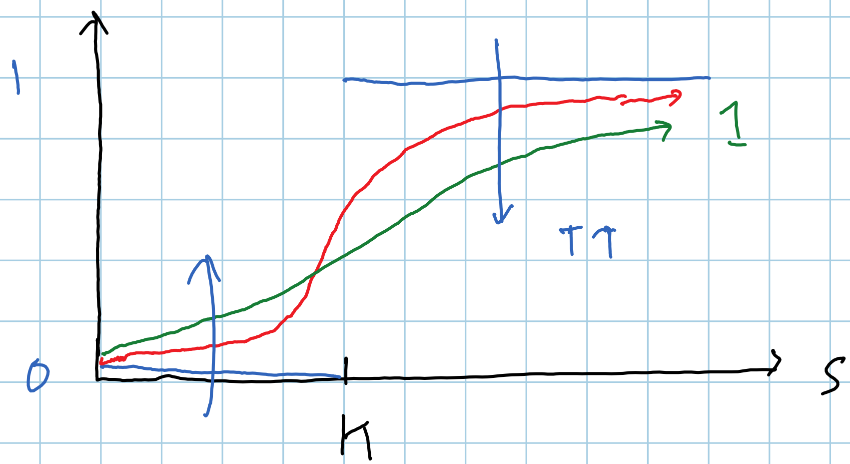
$1 \leftarrow \partial_S (S-K)_+$

$$= \left(\frac{1}{S}\right) \underbrace{e^{-r\tau}\, \mathbb{E}^{\mathbb{Q}}\left[S_T\, \mathbb{1}_{S_T > K} \right]}_{S\, \mathbb{E}^{\mathbb{Q}^S}\left[\mathbb{1}_{S_T > K}\right] = S\, \mathbb{Q}^S(S_T > K)}$$

$$= S\, \Phi(d_+)$$

$$S_T \overset{d}{=} e^{(r + \frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\, Z}, \quad Z \underset{\mathbb{Q}^S}{\sim} N(0,1)$$

$$\Rightarrow \quad \boxed{\Delta_t^C = \Phi(d_+)} \qquad d_+ = \frac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)\tau}{\sqrt{\sigma^2 \tau}}$$



recall $\quad C - P = S - Ke^{-r\tau} \quad$ (put-call parity)

$$\Delta^C - \Delta^P = 1$$

$$\Rightarrow \quad \Delta^P = \Delta^C - 1$$

$$\boxed{\Delta^P = -\Phi(-d_+)}$$

$$\Gamma = \partial_{SS}\, g = \partial_S \Delta$$

pdf of std. norm $\quad \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}$

$$\Gamma^C = \partial_S \Phi(d_+) = \phi(d_+)\, \partial_S d_+$$

$$= \frac{1}{S\sqrt{\sigma^2\tau}}\, \phi(d_+)$$

$\Gamma^p = \Gamma^c$