Dividend Paying Assets

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options on dividend paying assets.

- $\dfrac{dS_t}{S_t} = \mu(t, S_t)\, dt + \sigma(t, S_t)\, dW_t$ $\qquad \alpha_t$

  $\mu(t,S_t) \hookrightarrow \mu_t \qquad \sigma(t,S_t) \hookrightarrow \sigma_t$

  asset also pays continuous dividends: $\delta_t S_t \, dt$

  $D_t = \int_0^t \delta_u S_u \, du$ total dividends up to time t.

- $\dfrac{dB_t}{B_t} = r_t \, dt$ $\qquad \beta_t$

  $r_t \hookrightarrow r(t, S_t)$

- value a claim g has price $g_t = g(t, S_t)$ $\qquad -1$

  has payoff $\varphi(S_T)$

- $V_t = \alpha_t S_t + \beta_t B_t - g_t$
- $V_0 = 0$
- $dV_t = \alpha_t \, dS_t + \beta_t \, dB_t - dg_t + \alpha_t \delta_t S_t \, dt$

$$\underbrace{+ S_t\, d\alpha_t + B_t\, d\beta_t + d[\alpha, S]_t + d[\beta, B]_t}_{=0 \text{ self-financing}}$$

$d[\beta, B]_t \to 0$

$$= \alpha_t (\mu_t S_t\, dt + \sigma_t S_t\, dW_t)$$
$$+ \beta_t\, r B_t\, dt$$
$$- (\partial_t + \mu_t S_t \partial_s + \tfrac{1}{2} \sigma_t^2 S_t^2 \partial_{ss}) g_t\, dt$$
$$- \sigma_t S_t \partial_s g_t\, dW_t$$
$$+ \alpha_t \delta_t S_t\, dt$$

$$\boxed{\alpha_t = \partial_s g_t}$$ locally remove risk

$\Rightarrow dV_t = \{ \cdot \}\, dt$ to avoid arbitrage $\{ \cdot \} = 0$!

now since $V_0 = 0$ + $dV_t = 0 \Rightarrow V_t = 0\ \forall t$.

$$\boxed{\beta_t = (g_t - \alpha_t S_t) B_t^{-1}}$$

$\{ \cdot \} = 0$

$$\Rightarrow 0 = \alpha_t \mu_t S_t + r_t \beta_t B_t - (\partial_t + \mu_t S_t \partial_s + \tfrac{1}{2} \sigma_t^2 S_t^2 \partial_{ss}) g_t$$

$$+ \ \alpha_t \ \delta_t S_t$$

$$\Rightarrow \quad \left(\partial_t + (r_t - \delta_t) S_t \partial_s + \tfrac{1}{2} \sigma_t^2 S_t^2 \partial_{ss} \right) g_t = r_t \, g_t$$

holds $\forall$ paths

$\Rightarrow$

$$\boxed{\begin{aligned} \left(\partial_t + \left(r(t,S) - \delta(t,S) \right) S \partial_s + \tfrac{1}{2} \sigma^2(t,S) S^2 \partial_{ss} \right) g(t,S) &= r(t,S) \, g(t,S) \\ g(T,S) &= \varphi(S) \end{aligned}}$$

$$g(t,S) = \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^T r(u,S_u)\,du} \, \varphi(S_T) \;\middle|\; S_t = S \right]$$

$$\frac{dS_t}{S_t} = (r_t - \delta_t)\,dt + \sigma_t \, d\hat{W}_t$$

↳ $\mathbb{Q}$ - B.mtn.

Options on Futures

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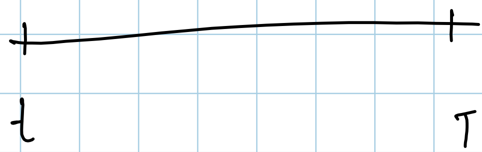
Forward contract



$$f_T(T) = S_T - K$$

value of forward contract

$$F_t(T) = \mathbb{E}^{Q}\left[e^{-\int_t^T r_u\,du} (S_T - K) \right]$$

$$= S_t - K\, P_t(T)$$

Forward price $F_t(T)$ is the strike s.t. forward contract has zero value at time t.

$$F_t(T) = \frac{S_t}{P_t(T)}$$

$$F_t(T) = \mathbb{E}^{Q}\left[e^{-\int_t^T r_u\,du} S_T \right] - K\, P_t(T)$$

$$= \mathbb{E}^{Q_T}[S_T]\, P_t(T) - K\, P_t(T)$$

$$\Rightarrow \boxed{F_t(T) = \mathbb{E}^{Q_T}[S_T]}$$

Futures contracts

- It costs 0 to enter / leave contract

- It costs 0 to enter / leave contract

- Gains / losses are paid by the players.

$F_t(T)$ – Futures price

$$100 \longrightarrow 98 \longrightarrow 99 \longrightarrow 103 \longrightarrow 101$$

ΔF		-2		$+1$		$+4$		-2	
$t=$	0		1		2		3		4

$$F_T(T) = S_T$$

- $$\frac{dF_t(T)}{F_t(T)} = \mu_t\, dt + \sigma_t\, dW_t \qquad \alpha_t$$

- $$\frac{dB_t}{B_t} = r(t, F_t(T))\, dt \qquad \beta_t$$

- claim $g_t = g(t, F_t(T))$, $g(T_0, F_{T_0}(T)) = \varphi(F_{T_0}(T))$

-1

$$V_t = \alpha_t \overset{0}{F_t(T)} + \beta_t B_t - g_t$$

$$dV_t = \beta_t\, dB_t - dg_t + \alpha_t\, dF_t$$

$\llcorner$ self-financing

$$= \beta_t \, r \, B_t - \left[(\partial_t + \mathcal{L}) g_t \, dt + \sigma_t F_t \partial_F g_t \, dW_t \right]$$

$$+ \alpha_t \mu_t F_t \, dt + \alpha_t \sigma_t F_t \, dW_t$$

$\alpha_t = \partial_F g_t$ removes local risk

$\Rightarrow \quad dV_t = \{\cdot\} \, dt$, to avoid arbitrage $\{\cdot\} = 0$!

since $V_0 = 0$ & $dV_t = 0 \Rightarrow V_t = 0$

$$\Rightarrow \quad \beta_t = B_t^{-1} g_t$$

$\{\cdot\} = 0$

$$\Rightarrow \quad r_t g_t - \left(\partial_t g_t + \mu_t F_t \partial_F g_t + \tfrac{1}{2} \sigma_t^2 F_t^2 \partial_{FF} g_t \right)$$

$$+ \alpha_t \mu_t F_t = 0$$

$\llcorner \partial_F g_t$

$$\Rightarrow \quad \partial_t g_t + \tfrac{1}{2} \sigma_t^2 F_t^2 \partial_{FF} g_t = r_t g_t$$

$\forall$ paths

$$\Rightarrow \boxed{\begin{cases} \partial_t g(t,F) + \frac{1}{2}\sigma^2(t,F)\, F^2\, \partial_{FF}\, g(t,F) = r(t,F)\, g(t,F) \\ g(T_0, F) = \varphi(F) \end{cases}}$$

$$\Rightarrow \quad g(t,F) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T_0} r(u, F_u)\, du} \cdot \varphi(F_{T_0}) \,\middle|\, F_t(T) = F \right]$$

$$\frac{dF_t}{F_t} = \sigma(t, F_t)\, d\widehat{W}_t$$

$F_t(T)$ is a $\mathbb{Q}$-mtg!

$$\Rightarrow \quad F_t(T) = \mathbb{E}^{\mathbb{Q}}[\, \underbrace{F_{u \to T}(T)}_{S_T} \,]$$

$$\boxed{F_t(T) = \mathbb{E}^{\mathbb{Q}}[S_T]}$$

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forward price $\quad G_t(T) = \mathbb{E}_t^{\mathbb{Q}^T}[S_T]$

Futures price $\quad F_t(T) = \mathbb{E}_t^{\mathbb{Q}}[S_T]$

Vasicek $dr_t = \kappa(\theta - r_t)\,dt + \eta\, dW_t^r$

$\frac{dS_t}{S_t} = r_t\,dt + \sigma\, dW_t^S$ $\Big\}$ $[W^r, W^S]_t = 0$

$$S_T = S_t \exp\left\{ \int_t^T (r_u - \tfrac{1}{2}\sigma^2)\,du + \sigma(W_T^S - W_t^S) \right\}$$

$$F_t(T) = S_t\, \mathbb{E}^{Q}\left[e^{\int_t^T r_u\,du} \right] \underbrace{\mathbb{E}^{Q}\left[e^{-\frac{1}{2}\sigma^2(T-t) + \sigma\sqrt{T-t}\,Z} \right]}_{1}$$

$$G_t(T) = \mathbb{E}^{Q^T}\left[S_T \right] = \mathbb{E}^{Q^T}\left[\frac{S_T}{P_T(T)} \right] = \frac{S_t}{P_t(T)}$$

$X_t = \frac{S_t}{P_t(T)}$ is a Q^T-mtg.

$$P_t(T)^{-1} \overset{?}{=} \mathbb{E}\left[e^{\int_t^T r_u\,du} \right]$$

$\parallel$

$$\left(\mathbb{E}\left[e^{-\int_t^T r_u du} \right] \right)^{-1} \qquad Y = e^{\int_t^T r_u\,du}$$

$$\left(\mathbb{E}\left[\tfrac{1}{Y} \right] \right)^{-1} = \mathbb{E}\left[Y \right]$$