FX Rates

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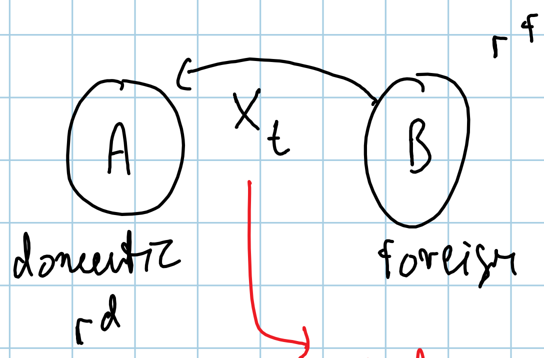
Exchange Rates:
(FX)



value of \$1 B in terms of A.

$\mathbb{P}$ - B.motion

$$\frac{dX_t}{X_t} = \mu_t \, dt + \sigma_t \, dW_t$$

$$\frac{dB_t^d}{B_t^d} = r^d \, dt \qquad \text{domestic bank account}$$

$$\frac{dB_t^f}{B_t^f} = r^f \, dt \qquad \text{foreign bank account}$$

$\tilde{B}_t^f = (X_t \, B_t^f)$ ~ Foreign bank acct in domestic \$
(domestic traded asset)

domestic assets grow at the domestic risk-free rate under the domestic risk-neutral measure
($\mathbb{Q}^d$)

rate under the domestic risk-neutral measure ($\mathbb{Q}^d$)

moreover domestic assets (discounted at the domestic r^d) are $\mathbb{Q}^d$-mtg.

$\iff$ no arb.

$$d\tilde{B}^f_t = d(B^f_t X_t) = dB^f_t \, X_t + B^f_t \, dX_t + \underbrace{d[X, B^f]_t}_{0}$$

$$= r^f B^f_t X_t \, dt + B^f_t (\mu_t X_t \, dt + \sigma_t X_t \, dW_t)$$

$\Rightarrow$

$$\frac{d\tilde{B}^f_t}{\tilde{B}^f_t} = (r^f + \mu_t)\, dt + \sigma_t \, dW_t$$

$$= r^d \, dt + \sigma_t \underbrace{\Big(\overbrace{\frac{-r_d + r^f + \mu_t}{\sigma_t}}^{\lambda^d_t} dt + dW_t \Big)}_{dW^d_t}$$

Girsanov Thm says: $\exists \; \mathbb{Q}^d$ s.t. $W^d_t = \int_0^t \lambda^d_u \, du + W_t$

is a $\mathbb{Q}^d$-B. motn.

specifically

$$\left(\frac{d\mathbb{Q}^d}{d\mathbb{P}}\right) = \mathcal{E}\left(-\int_0^\infty \lambda^d_u \, dW_u\right)$$

$$\left(\frac{d\mathbb{Q}^d}{d\mathbb{P}}\right) = \mathcal{E}\left(\quad 0 \quad \right)$$

$$= \exp\left\{ -\frac{1}{2}\int_0^\infty (\lambda_u^d)^2\, du - \int_0^\infty \lambda_u^d\, dW_u \right\}$$

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so $$\frac{dX_t}{X_t} = \mu_t\, dt + \sigma_t \underbrace{\left(-\lambda_t^d\, dt + dW_t^d\right)}_{dW_t}$$

$$= (\mu_t - \lambda_t^d \sigma_t)\, dt + \sigma_t\, dW_t^d$$

$$= (r^d - r^f)\, dt + \sigma_t\, dW_t^d$$

$$\boxed{\frac{dX_t}{X_t} = (r^d - r^f)\, dt + \sigma_t\, dW_t^d}$$

looks as if X_t is a dividend paying asset with dividends of r^f.

Suppose $\sigma_t = \text{const}$.

i) value a call on the FX rate:

$$\mathcal{C} = (X_T - F)_+ \quad \text{in } \#A$$

$$V_0 = \mathbb{E}^{\mathbb{Q}^d}\left[e^{-r^d T} \cdot (X_T - F)_+ \right]$$

$$dX_t = (r^d - r^f)\, dt + \sigma\, dW_t^d$$

$$\frac{dX_t}{X_t} = (r^d - r^f)\,dt + \sigma\, dW_t^d$$

$$\rightarrow \quad X_T = X_0\, e^{((r^d - r^f) - \frac{1}{2}\sigma^2)T + \sigma W_T^d}$$

$$V_0 = e^{-r_f T}\, \mathbb{E}^{\mathbb{Q}^d}\left[e^{-(r^d - r^f)T} (X_T - F)_+ \right]$$

$$= e^{-r_f T}\left(X_0\, \Phi(d_+) - F \cdot e^{-(r^d - r^f)T}\, \Phi(d_-) \right)$$

$$d_\pm = \frac{\ln(X_0/F) + ((r^d - r^f) \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

ii) $\quad \varphi = (X_T - F)_+ \quad$ in $\cancel{\$}\,\mathcal{B} \quad @\ T$

$$V_0 = \mathbb{E}^{\mathbb{Q}^d}\left[e^{-r_d T} (X_T - F)_+ \cdot X_T \right]$$

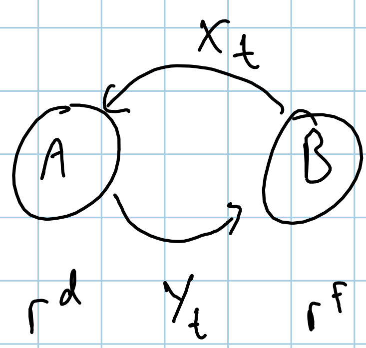
try to value using $\tilde{B}_t^f$ as a numeraire asset

$$\frac{V_0}{\tilde{B}_0^f} = \mathbb{E}^{\tilde{\mathbb{Q}}^f}\left[\frac{(X_T - F)_+ \cancel{X_T}}{\tilde{B}_T^f} \right]$$

$\hookrightarrow \quad e^{r^f T} \cdot \cancel{X_T}$

. . .

foreign risk-neutral measure



$\hookrightarrow 1/X_t$ $\qquad Y_t = f(X_t)$, $f(x) = 1/x$

$$dY_t = \left(0 + \mu_t X_t \cdot \left(-\frac{1}{X_t^2}\right) + \frac{1}{2}\sigma_t^2 X_t^2 \cdot \frac{2}{X_t^3}\right) dt$$

$$+ \sigma_t X_t \cdot \left(-\frac{1}{X_t^2}\right) dW_t$$

$$\boxed{\frac{dY_t}{Y_t} = \left(-\mu_t + \sigma_t^2\right) dt - \sigma_t \, dW_t}$$

$\tilde{B}_t^d = (B_t^d Y_t)$ is value of a <u>foreign</u> traded asset

$$\frac{d\tilde{B}_t^d}{\tilde{B}_t^d} = \left(r^d - \mu_t + \sigma_t^2\right) dt - \sigma_t \, dW_t$$

$$= r^f \, dt - \sigma_t \left[\frac{r^f - r^d + \mu_t - \sigma_t^2}{\sigma_t} dt + dW_t \right]$$

$$\underbrace{\qquad\qquad}_{\lambda^f_t}$$

Girsanov's Thm: $W^f_t = \int_0^t \lambda^f_u \, du + W_t$ is a $\mathbb{Q}^f$ - B.mtn.

$$\left(\frac{d\mathbb{Q}^f}{d\mathbb{P}}\right) = \exp\left\{ -\frac{1}{2}\int_0^\infty (\lambda^f_u)^2 \, du - \int_0^\infty \lambda^f_u \, dW_u \right\}$$

$\mathbb{Q}^f$ is the foreign risk-neutral measure.

$$\frac{dY_t}{Y_t} = (-\mu_t + \sigma_t^2)\,dt - \sigma_t\left[-\lambda^f_t \, dt + dW^f_t \right]$$

$$= (-\underline{\mu_t + \sigma_t^2} + \underbrace{\sigma_t \lambda^f_t}_{(r^f - r^d + \mu_t - \sigma_t^2)})\,dt - \sigma \, dW^f_t$$

$$\boxed{\frac{dY_t}{Y_t} = (r^f - r^d)\,dt - \sigma \, dW^f_t}$$

exchange rate looks like a foreign asset paying dividend of r^d

$$
\begin{tikzcd}
 & \mathbb{P} \arrow[dl, "\lambda^d_t"'] \arrow[dr, "\lambda^f_t"] & \\
\mathbb{Q}^d \arrow[rr] & & \mathbb{Q}^f
\end{tikzcd}
$$

$$dW^d_t = \lambda^d_t \, dt + dW_t$$

$$dW^f_t = \lambda^f_t \, dt + dW_t$$

$$dW^d_t = (\lambda^d_t - \lambda^f_t)\, dt + dW^f_t$$

$$\lambda^d_t = \frac{r^f - r^d + \mu_t}{\sigma_t} \qquad \lambda^f_t = \frac{r^f - r^d + \mu_t - \sigma_t^2}{\sigma_t}$$

$$\Rightarrow \quad \boxed{dW^d_t = \sigma_t \, dt + dW^f_t}$$

$$\frac{dX_t}{X_t} = (r^d - r^f)\, dt + \sigma_t \, dW^d_t$$

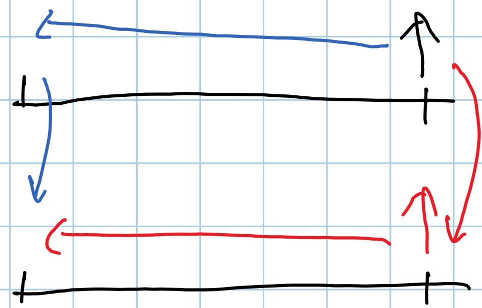
$$= (r^d - r^f + \sigma_t^2)\, dt + \sigma_t \, dW^f_t$$

to value $(X_T - K)_+$ in \$ B @ T

$\mathbb{E}^{\mathbb{Q}^f}[e^{-r^f T} \ell] \cdot X_0 \longleftarrow$

$\mathbb{E}^{Q^f}[e^{-r^f T} Q] \cdot X_0$

$\mathbb{E}^{Q^d}[e^{-r^d T} Q \cdot X_T]$



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Foreign Asset

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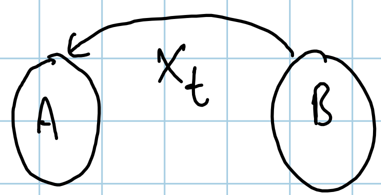

r^d $\qquad\qquad$ r^f, S_t

$$\frac{dX_t}{X_t} = \mu_t\, dt + \sigma_t\, dW_t \qquad\qquad d[B,W]_t = \rho\, dt$$

$$\frac{dS_t}{S_t} = \nu_t\, dt + \eta_t\, dB_t$$

we already know that

1) $$\frac{dS_t}{S_t} = r^f\, dt + \eta_t\, dB_t^f$$

$\hookrightarrow$ Foreign risk-neutral measure.

2) $$dW_t^d = \sigma_t\, dt + dW_t^f$$

$$B_t = \rho W_t + \sqrt{1-\rho^2}\, W_t^{\perp}$$

$\llcorner$ independent of W_t

$$dB_t^f = \rho\, dW_t^f + \sqrt{1-\rho^2}\, dW_t^{\perp f}$$

$$
\begin{aligned}
dB_t &= \rho\, dW_t + \sqrt{1-\rho^2}\, dW_t \\
&= \rho\left(-\sigma_t\, dt + dW_t^d\right) + \sqrt{1-\rho^2}\, dW_t^{\perp f} \\
&= -\rho\sigma_t\, dt + \rho\, dW_t^d + \sqrt{1-\rho^2}\left(0\, dt + dW_t^{\perp d}\right) \\
&= -\rho\sigma_t\, dt + \left(\rho\, dW_t^d + \sqrt{1-\rho^2}\, dW_t^{\perp d}\right) \rightsquigarrow dB_t^d
\end{aligned}
$$

$$
\begin{tikzcd}
 & \mathbb{P} \arrow[dl, "\lambda_t^{d,x}"'] \arrow[dr, "\lambda_t^{f,x}"] & \\
\mathbb{Q}^d \arrow[rr, "\text{“}W\text{”}"'] & & \mathbb{Q}^f
\end{tikzcd}
$$

“W” $\lambda_t^{d,x}$ $\quad$ $\lambda_t^{f,x}$ “W”

$\lambda_t^{f,S}$ “B” $\rightarrow$ “W” “$W^{\perp}$”

$$
\boxed{dB_t^f = -\rho\sigma_t\, dt + dB_t^d}
$$

$$
\Rightarrow \quad \frac{dS_t}{S_t} = r^f\, dt + \eta_t\left(-\rho\sigma_t\, dt + dB_t^d\right)
$$

$$
\boxed{\frac{dS_t}{S_t} = \left(r^f - \rho\sigma_t\eta_t\right) dt + \eta_t\, dB_t^d}
$$

is not a domestic traded asset of any kind.

$\tilde{S}_t = S_t X_t$ is value of a domestic asset

$$d\tilde{S}_t = dS_t\, X_t + S_t\, dX_t + d[S,X]_t$$

$$= (\nu_t\,dt + \eta_t\,dB_t) \cdot S_t X_t$$

$$+ S_t(\mu_t\,dt + \sigma_t\,dW_t)\, X_t$$

$$+ \rho\,\sigma_t \eta_t\, S_t X_t\, dt$$

$\rho W_t + \sqrt{1-\rho^2}\, W_t^{\perp}$

$$\frac{d\tilde{S}_t}{\tilde{S}_t} = (\nu_t + \mu_t + \rho\,\sigma_t\eta_t)\,dt + \eta_t\,dB_t + \sigma_t\,dW_t$$

$$= r^d\,dt + \sigma_t\Big(\underbrace{\lambda_t^d dt + dW_t}_{dW_t^d}\Big)$$

$\lambda_t^d = \dfrac{r^f - r^d + \mu_t}{\sigma_t}$

$$+ \eta_t\Big(\underbrace{\text{---} + dB_t}_{dB_t^d}\Big)$$

Examples

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$$\$B$$

i) $\quad \varphi = \underset{\swarrow}{(S_T - K)_+} \text{ in } \$B \text{ @ } T$

$$V_0 = X_0 \, \mathbb{E}^{\mathbb{Q}^f} \left[(S_T - K)_+ e^{-r^f T} \right]$$

$$= X_0 \left(S_0 \, \Phi(d_+) - K e^{-r^f T} \Phi(d_-) \right)$$

$$d_\pm = \frac{\ln(S_0/K) + (r^f \pm \frac{1}{2}\eta^2)T}{\eta \sqrt{T}}$$

hard approach:

$$V_0 = \mathbb{E}^{\mathbb{Q}^d} \left[(S_T - K)_+ \cdot X_T \, e^{-r^d T} \right]$$

we have that

$$\frac{dS_t}{S_t} = (r^f - \rho \sigma \eta) dt + \eta \, dB_t^d$$

$$\frac{dX_t}{X_t} = (r^d - r^f) dt + \sigma \, dW_t^d$$

$$\tilde{M}^f_t = X_t \, e^{r^f t},$$

$$\frac{d\tilde{M}^f_t}{\tilde{M}^f_t} = r^d \, dt + \sigma \, dW^d_t$$

$$\frac{V_0}{\tilde{M}^f_0} = \mathbb{E}^{\tilde{\mathbb{Q}}^f}\left[\frac{(S_T - K)_+ \cdot X_T}{X_T \, e^{r^f T}} \right]$$

$$= \mathbb{E}^{\tilde{\mathbb{Q}}^f}\left[(S_T - K)_+ \right] e^{-r^F T}$$

$$\mathfrak{z}_t = \left(\frac{d\tilde{\mathbb{Q}}^f}{d\mathbb{Q}^d} \right)_t = \frac{\tilde{M}^f_t / \tilde{M}^f_0}{M^d_t / M^d_0} = e^{(r^f - r^d) t} \, X_t$$

$$\Rightarrow \frac{d\mathfrak{z}_t}{\mathfrak{z}_t} = \sigma \, dW^d_t + 0 \, dW^{\perp d}_t$$

Girsanov $\Rightarrow$ $d\tilde{W}^f_t = -\sigma \, dt + dW^d_t$

note this is identical to $\mathbb{Q}^f$!

$$\Rightarrow d\tilde{B}^f_t = -\rho\sigma \, dt + dB^d_t$$

$$\frac{dS_t}{S_t} = (r^f - \rho\sigma\eta)\,dt + \eta\, dB_t^d$$

$$= r^f\, dt + \eta\, d\tilde{B}_t^f$$

$$\frac{V_0}{\tilde{M}_0^f} = e^{-r^f T}\, \mathbb{E}^{\tilde{Q}^f}\left[(S_T - K)_+ \right]$$

$\hookrightarrow X_0$

Examples

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"spot FX rate"

$\$B$

iii) $\quad \varphi = (S_T - K)_+ \, x \quad$ in $\$A$ @ T

$$\frac{dS_t}{S_t} = \underbrace{(r^f - \rho\sigma\eta)}_{\sim \hat{r}}\,dt + \eta\, dB_t^d$$

$$V_0 = \mathbb{E}^{\mathbb{Q}^d}\left[(S_T - K)_+ \, x \, e^{-r^d T}\right]$$

$$= e^{(\hat{r} - r^d)T} \cdot \underbrace{e^{-\hat{r}T}\,\mathbb{E}^{\mathbb{Q}^d}\left[(S_T - K)_+\right]}_{S_0\,\Phi(d_+) - K e^{-\hat{r}T}\,\Phi(d_-)} \, x$$

$$d_\pm = \frac{\ln(S_0/K) + (\hat{r} \pm \frac{1}{2}\eta^2)T}{\eta\sqrt{T}}$$