Interest Rate Caps

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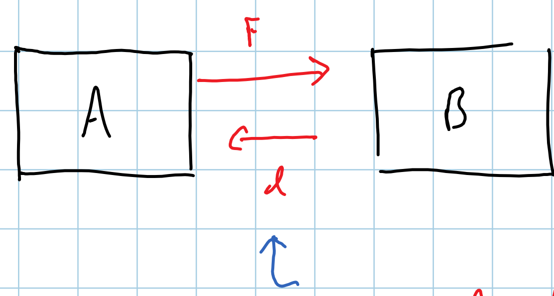
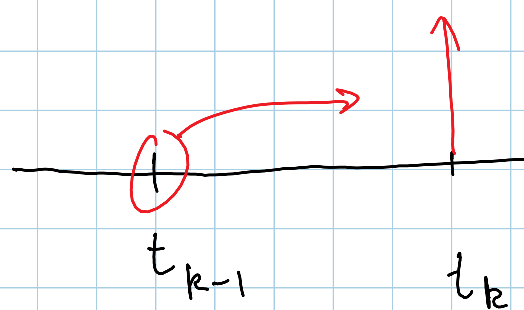
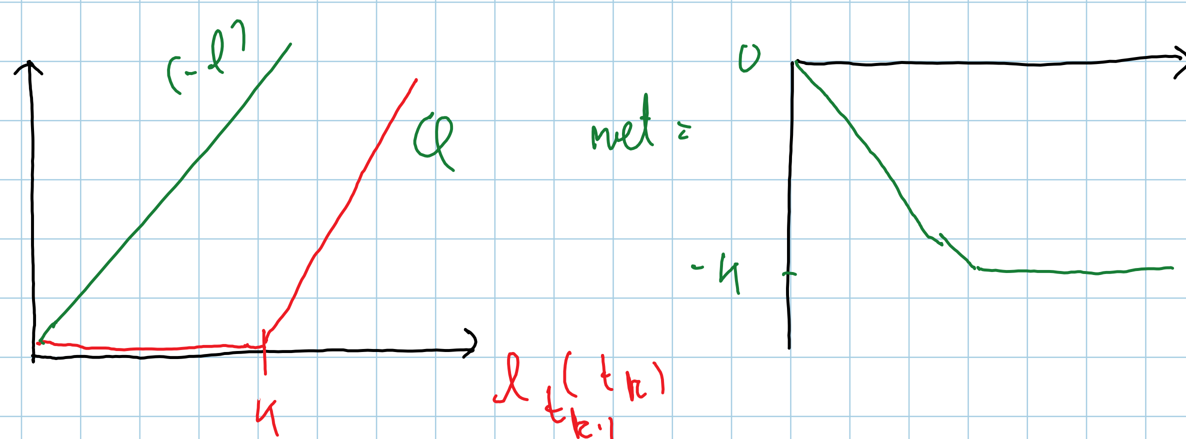
Interest Derivatives – linked to IRS



$l_{t_{k-1}}(t_k)\,\Delta t_k$



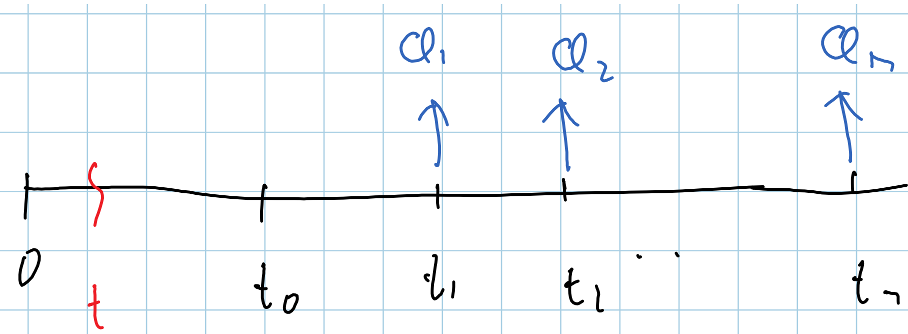
$l_t^{(k)} \triangleq l_t(t_k)$



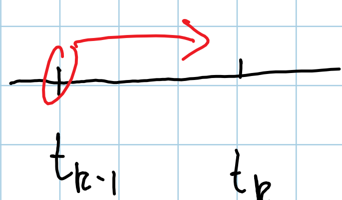
caplets

Cap = collection of caplets at each payment date

$Cl_1 \qquad Cl_2 \qquad Cl_n$



$$V_t^{(k)} = \mathbb{E}^{\mathbb{Q}}\left[\left(L^{(k)}_{t_{k-1}} - K \right)_+ e^{-\int_t^{t_k} r_s\, ds} \right]$$



$$\left(1 + \Delta t_k L^{(k)}_{t_{k-1}}\right)^{-1} = P_{t_{k-1}}(t_k)$$

$$\Rightarrow \quad L^{(k)}_{t_{k-1}} = \frac{1}{\Delta t_k}\left[\frac{1}{P_{t_{k-1}}(t_k)} - 1 \right]$$

$$\Rightarrow \quad \frac{V_t^{(k)}}{P_t(t_k)} = \mathbb{E}^{\mathbb{Q}^k}\left[\frac{\left(L^{(k)}_{t_{k-1}} - K \right)_+}{\underbrace{P_{t_k}(t_k)}_{\to 1}} \right]$$

$$x_t^{(k)} = \frac{P_t(t_{k-1})}{P_t(t_k)}$$

Moreover, $x^{(k)}_{t_{k-1}} = \frac{1}{\quad}$

moreover, $x^{(k)}_{t_{k-1}} = \frac{1}{P_{t_{k-1}}(t_k)}$

$$\Rightarrow \quad l^{(k)}_t = \frac{1}{\Delta t_k}\left(\frac{P_t(t_{k-1})}{P_t(t_k)} - 1\right) = \frac{1}{\Delta t_k}\left(x^{(k)}_t - 1\right)$$

$$\xrightarrow[t \uparrow t_{k-1}]{} l^{(k)}_{t_{k-1}}$$

$$\Rightarrow \quad \frac{V^{(k)}_t}{P_t(t_k)} = \mathbb{E}^{\mathbb{Q}^k}\left[\left(\frac{1}{\Delta t_k}\left(x^{(k)}_{t_{k-1}} - 1\right) - K\right)_+\right]$$

$$= \frac{1}{\Delta t_k}\mathbb{E}^{\mathbb{Q}^k}\left[\left(x^{(k)}_{t_{k-1}} - \underbrace{(1 + \Delta t_k K)}_{\alpha}\right)_+\right]$$

$$\frac{d x^{(k)}_t}{x^{(k)}_t} = \left(\mathrm{vol}\ t_{k-1} - \mathrm{vol}\ t_k\right) dW^{(k)}_t$$

$-\sigma B_t(t_{k-1}) \qquad -\sigma B_t(t_k)$

(Vasicek model)

$\mathbb{Q}^{(k)}$ B. motion

Σ_t

$$d \ln x^{(k)}_t = -\frac{1}{2}\Sigma^2_t\, dt + \Sigma_t\, dW^{(k)}_t$$

$$\ln x^{(k)}_{t_{k-1}} - \ln x^{(k)}_t = -\frac{1}{2}\int_t^{t_{k-1}} \Sigma^2_u\, du + \int_t^{t_{k-1}} \Sigma_u\, dW^{(k)}_u$$

$$\Rightarrow \quad x^{(k)}_{t_{k+1}} = x^{(k)}_t \exp\left\{ -\frac{1}{2} \underbrace{\int_t^{t_{k+1}} \Sigma_u^2 \, du}_{\Omega^2} + \underbrace{\int_t^{t_{k+1}} \Sigma_u \, dW_u^{(k)}}_{X} \right\}$$

$$X \underset{\mathbb{Q}^{(k)}}{\sim} \mathcal{N}\Big(0 \, ; \underbrace{\int_t^{t_{k+1}} \Sigma_u^2 \, du}_{\Omega^2} \Big)$$

$$\mathbb{E}^{\mathbb{Q}^k}\left[\left(x^{(k)}_{t_{k+1}} - \alpha \right)_+ \right]$$

$$= x^{(k)}_t \, \Phi(d_+^{(k)}) - \alpha^{(k)} \Phi(d_-^{(k)})$$

$$d_\pm^{(k)} = \frac{\ln\left(x_t^{(k)} / \alpha^{(k)}\right) \pm \frac{1}{2} (\Omega^{(k)})^2}{\Omega^{(k)}}$$

$$\boxed{V_t^{(k)} = \frac{P_t^{(k)}}{\Delta t_k} \left[x_t^{(k)} \, \Phi(d_+^{(k)}) - \alpha^{(k)} \, \Phi(d_-^{(k)}) \right]}$$

IR Cap Implied Vols

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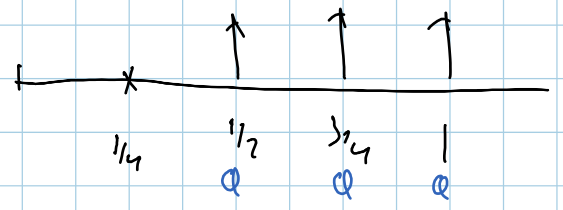
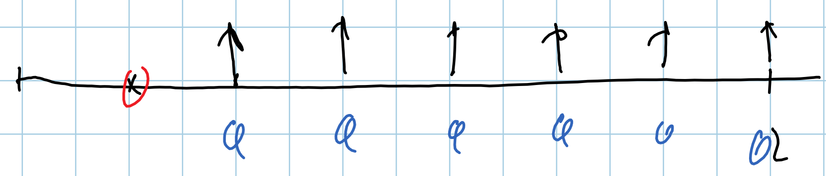
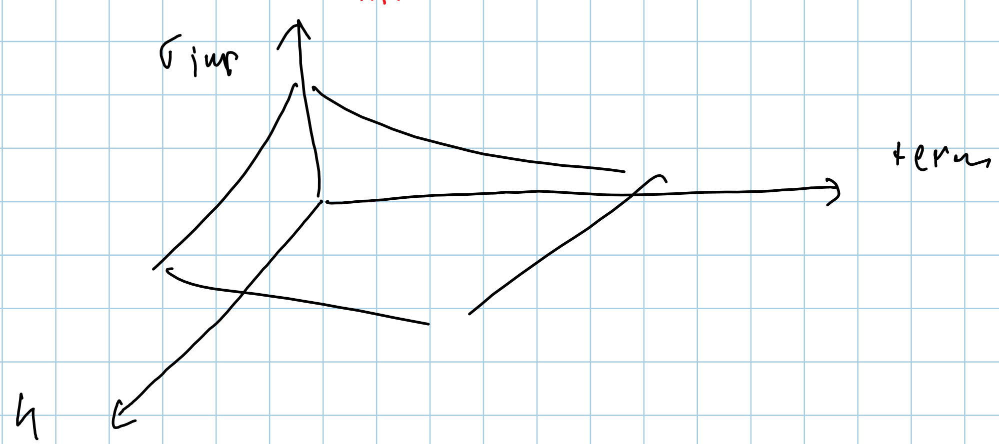


$K = (0.6, 0.7, .. 1.4)\, K_{ATM}(1)$

σ imp



$K = \{0.6, .., 1.4\}\, K_{ATM}(2)$



$$l_t^{(k)} = \frac{1}{\Delta t_k}\left[\frac{P_t(t_{k-1})}{P_t(t_k)} - 1\right]$$

$$l_{t_{k-1}}^{(k)} \overset{d}{=} l_t^{(k)} \cdot e^{-\frac{1}{2}\sigma^2 (t_{k-1}-t) + \sigma\sqrt{t_{k-1}-t}\; Z}$$

$$Z \sim N(0,1)$$

$$V_t^{(k)} = P_t(t_k)\, \underbrace{\mathbb{E}\left[\left(l_{t_{k-1}}^{(k)} - K\right)_+\right]}$$

$$l_t^{(k)}\, \Phi\left(d_+^{(k)}\right) - K\, \Phi\left(d_-^{(k)}\right)$$

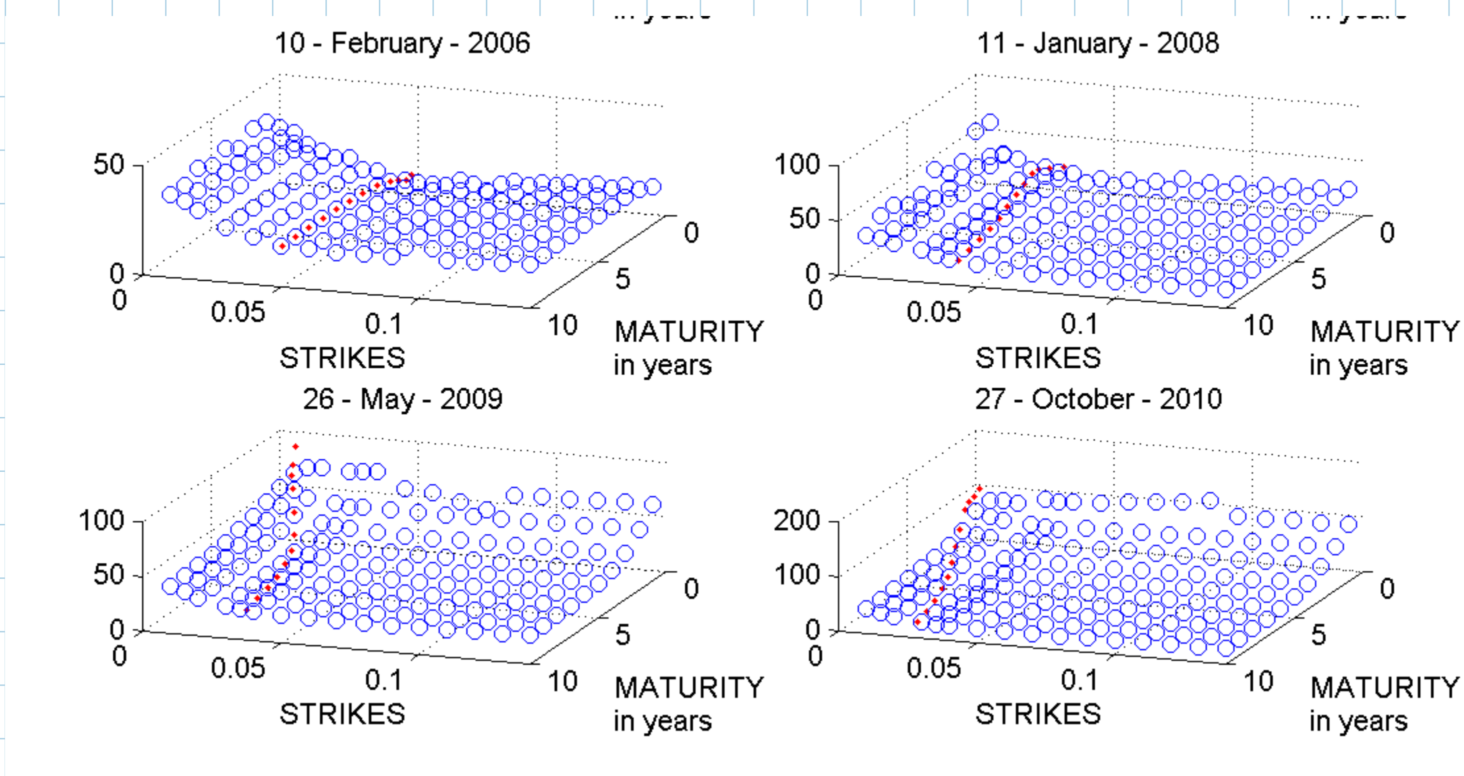
$$d_\pm^{(k)} = \frac{\ln\left(l_t^{(k)}/K\right) \pm \frac{1}{2}\sigma^2 (t_{k-1} - t)}{\sigma \sqrt{t_{k-1} - t}}$$

Flat vols → $\sigma \quad \sigma \quad \sigma$

→ forward vols $\sigma_1 \quad \sigma_2 \quad \sigma_3$

$\sigma \quad \sigma \quad \sigma \quad \sigma \quad \sigma \quad \sigma \quad \sigma$

$\sigma_1 \quad \sigma_2 \quad \sigma_3$



Swaptions

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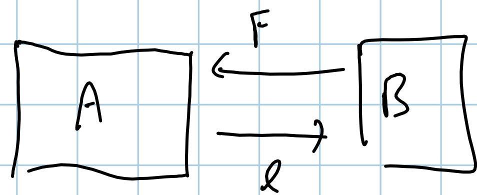
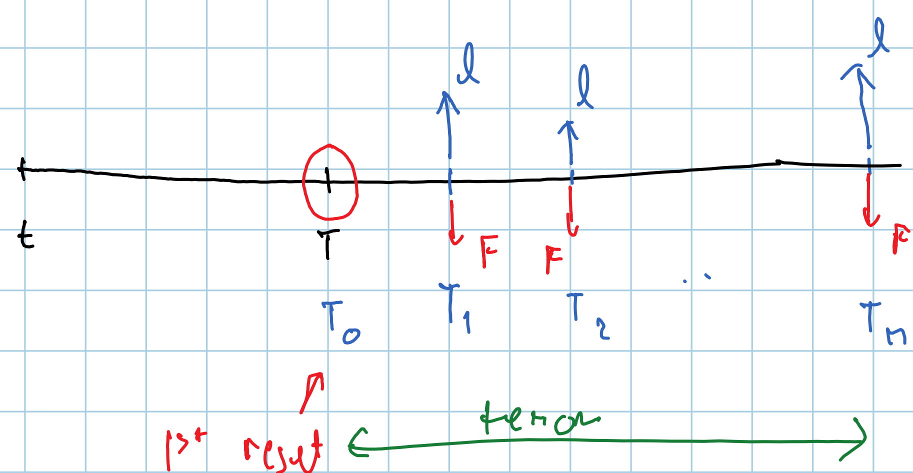
Swap Options



B enters into a payer swaption
mean they have the option to enter
into the payer side of an IRS
@ T with tenor T_1



$$V_T = \left(V_T^{fl} - V_T^{fix} \right)_+$$

$$= \left(P_T(T_0) - P_T(T_n) - F \sum_{k=1}^{n} \Delta t_k P_T(T_k) \right)_+$$

$$= \Big(\underbrace{\sum_{k=1}^{n} \Delta t_k P_T(T_k)}_{A_T} \Big) \Big(\boxed{\frac{P_T(T_0) - P_T(T_n)}{A_T}} - F \Big)_+$$

↳ swap-rate at T $S(T)$!

$$V_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} A_T (S_T - F)_+ \right]$$

$$= A_t \mathbb{E}^{\mathbb{Q}^A}\left[\frac{A_T (S_T - F)_+}{A_T} \right]$$

$$= A_t \mathbb{E}^{\mathbb{Q}^A}\left[(S_T - F)_+ \right]$$

$$S_t = \frac{P_t(T_0) - P_t(T_n)}{A_t}$$ is a $\mathbb{Q}^A$-mtg!

$$\frac{dS_t}{S_t} = \sigma_t \, dW_t^A$$ most always hold!

↳ deterministic assumption is called LSM

constant → Black Model

$$V_t = A_t \left[S_t \Phi(d_+) - F \Phi(d_-) \right]$$

$$d_\pm = \ln(S_t/F) \pm \frac{1}{2} \overbrace{\int_t^T \sigma_s^2 ds}^{\Sigma^2}$$

$$d_{\pm} = \frac{\ln(S_t/F) \pm \frac{1}{2}\left(\int_t^T \sigma_s^2\, ds\right)}{\Omega}$$

typically $F = S_t$ i.e. set ATM.

Swaptions Cont'd

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$T = T_0$.

$$V_T = \left(V_T^{Fl} - V_T^{fix} \right)_+$$

$$= \left(\left(1 - P_T(T_n)\right) - F \sum_{k=1}^{n} \Delta t_k P_T(T_k) \right)_+$$

$$= \Big(\underbrace{1 - \sum_{k=1}^{n} c_k P_T(T_k)}_{\text{Coupon Bearing Bond}} \Big)_+ \qquad c_k = \begin{cases} F \Delta t_k, & k < n \\ (1 + F \Delta t_k), & k = n \end{cases}$$

$$V_t = \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^T r_s ds} \left(1 - \sum_{k=1}^{n} c_k P_T(T_k) \right)_+ \right]$$

$$= P_t(T)\, \mathbb{E}^{\mathbb{Q}^T} \left[\left(1 - \frac{\sum_{k=1}^{n} c_k P_T(T_k)}{\underbrace{P_T(T)}_{\to 1}} \right)_+ \right]$$

assume affine model: $P_t(T_k) = e^{A_t(T_k) - B_t(T_k) r_t}$

this monotonically decreases in r_t so define

r^* s.t.

$$1 - \sum_{k=1}^{n} c_k P(T, r^*; T_k) = 0$$

then $\forall\, r > r^*$ option is in the money

then $\forall\, r > r^*$ option is in the money
(b/c $P \searrow$ in r)

$$\varphi = \Big(1 - \sum_k c_k P(T, r_T; T_k)\Big)_+$$

$$= \Big(1 - \sum_k c_k P(T, r_T; T_k)\Big) \mathbb{1}_{r_T > r^*}$$

$$= \underbrace{\mathbb{1}_{r_T > r^*}} - \sum_k c_k P(T, r_T; T_k) \underbrace{\mathbb{1}_{r_T > r^*}}_{\mathbb{1}_{P(T, r_T; T_k) < P(T, r^*; T_k)}}$$

$$\mathbb{1}_{r_T > r^*} = \mathbb{1}_{\underbrace{e^{A_T(T_k) - B_T(T_k) r_T}} < \underbrace{e^{A_T(T_k) - B_T(T_k) r^*}}}$$

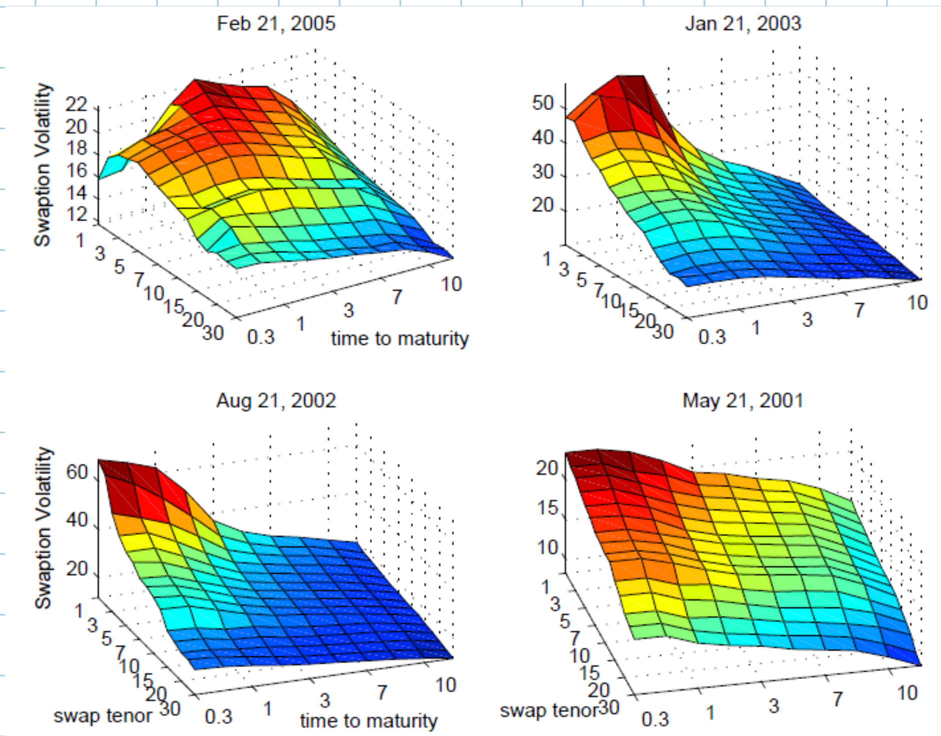
$$\varphi = \sum_k c_k P(T, r^*; T_k) \mathbb{1}_{(P(T, r_T; T_k) < P(T, r^*; T_k))}$$

$$- \sum_k c_k P(T, r_T; T_k) \mathbb{1}_{(\quad)}$$

D_k

$$= \sum_k c_k \big(P(T, r^*; T_k) - P(T, r_T; T_k)\big) \mathbb{1}_{D_k}$$

$$= \sum_k c_k \big(P(T, r^*; T_k) - P(T, r_T; T_k)\big)_+$$



Feb 21, 2005
Jan 21, 2003
Aug 21, 2002
May 21, 2001
Swaption Volatility
time to maturity
swap tenor