Binomial Model

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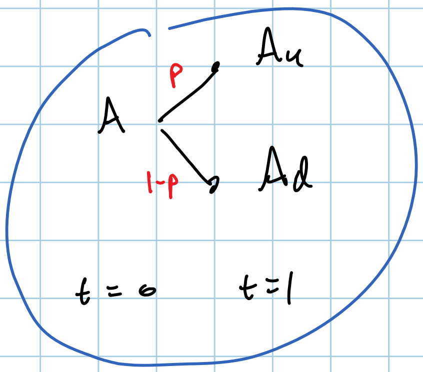


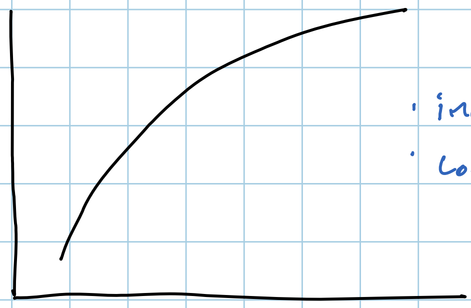
$$A \overset{?}{=} \frac{1}{1+r} \mathbb{E}[A_1]$$

$A_u = -A_d = 1$ $p = \frac{1}{2}$

$A_u = -A_d = 1000$

$u(x) : \mathbb{R} \mapsto \mathbb{R}$

utility Fn



- increasing
- concave

$$X_1 \preceq X_2 \iff \mathbb{E}[u(X_1)] \leq \mathbb{E}[u(X_2)]$$

prefer X_2 to X_1

e.g. $u(x) = -e^{-\gamma x}$ (exponential utility)

X_1 - no play x

X_2 - play $x - A + A_1$

$x + A - A_1$

indifference price A s.t. $\mathbb{E}[u(X_1)] = \mathbb{E}[u(X_2)]$

$$\Rightarrow \boxed{A^{buy} = -\frac{1}{\gamma} \ln \mathbb{E}[e^{-\gamma A_1}]} = -\frac{1}{\gamma} \ln\left(\mathbb{E}[e^{-\gamma(A_1 - A_d)}]\, e^{-\gamma A_d}\right)$$

$$A = \mathbb{E}[A_1]$$

$$= A_d - \frac{1}{\gamma} \ln \mathbb{E}[e^{-\gamma(A_1 - A_d)}]$$

$$A^{buy} \xrightarrow[\gamma \uparrow +\infty]{} A_d$$

$$\underset{\hookrightarrow +\infty}{\frac{1}{\gamma}} \ln\left(p \underset{\hookrightarrow 0}{e^{-\gamma(A_u - A_d)}} + (1-p)\right) \longrightarrow 0 .$$

$$\boxed{A^{sell} = +\frac{1}{\gamma} \ln \mathbb{E}[e^{\gamma A_1}]}$$

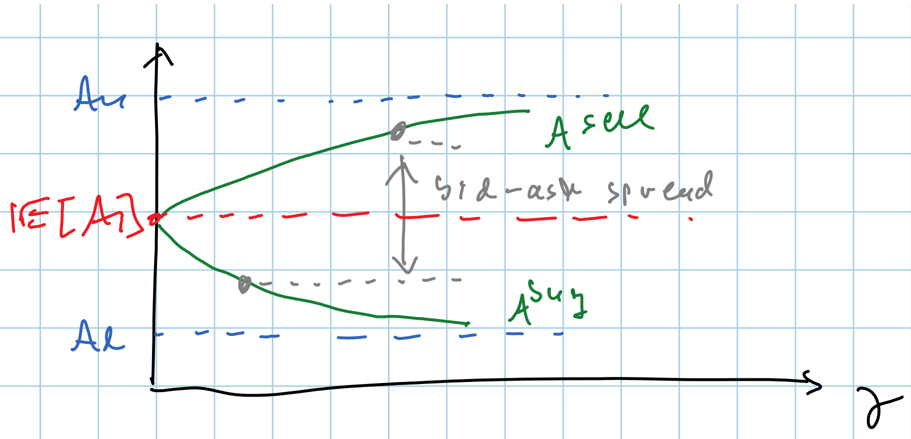
$$A^{sell} \xrightarrow[\gamma \uparrow +\infty]{} A_u$$

$$\mathbb{E}[e^{\gamma A_1}] = 1 + \underline{\gamma \mathbb{E}[A_1]} + o(\gamma)$$

$$\ln(1+z) = z + o(z)$$

$$\Rightarrow A^{sell} \xrightarrow[\gamma \downarrow 0]{} \mathbb{E}[A_1]$$

$$A^{buy} \xrightarrow[\gamma \downarrow 0]{} \mathbb{E}[A_1]$$


A_u
A^{sell}
bid-ask spread
$\mathbb{E}[A]$
A^{buy}
A_l
γ

Two Assets

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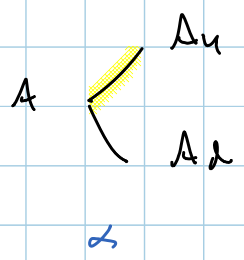
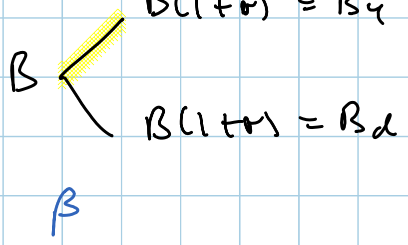
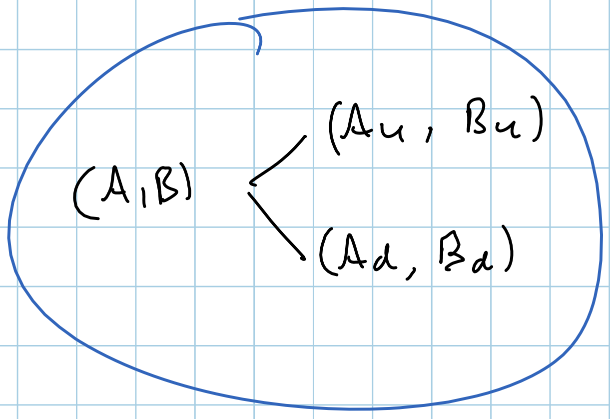
$A_u > A_d$







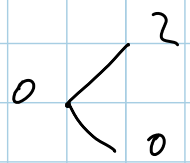
$$V_i = \alpha A_i + \beta B_i$$

an arbitrage portfolio (α, β) is one s.t.

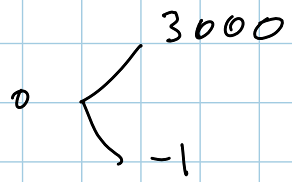
i) $V_0 = 0$ cost nothing

ii) $\exists$ a t s.t. a) $\mathbb{P}(V_t \geq 0) = 1$ never lose

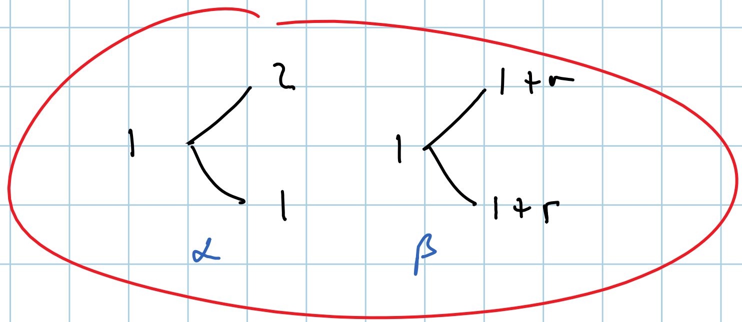
b) $\mathbb{P}(V_t > 0) > 0$ sometimes win



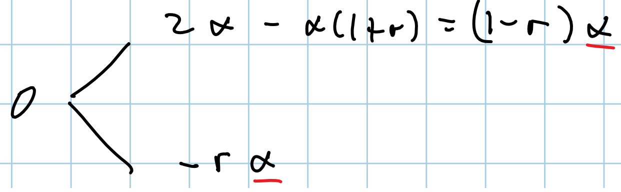
is an arb



is not an arb



$V_0 = 0 \Rightarrow \alpha = -\beta$:

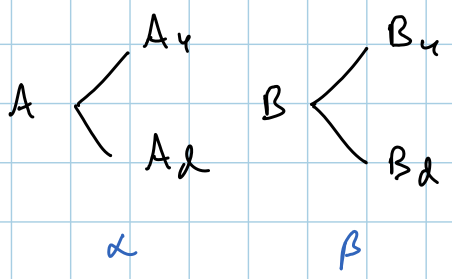


$1-r > 0$ & $-r > 0$ ⇒ $r \underline{\le} 0$ ∃ arb.

$1-r < 0$ & $-r < 0$ ⇒ $r \underline{\ge} 1$ ∃ arb.

to avoid arb need $\left.\begin{array}{ll} 1-r > 0, & -r < 0 \\ \text{or } 1-r < 0, & -r > 0 \end{array}\right\}$

⇒ $r \in (0,1)$

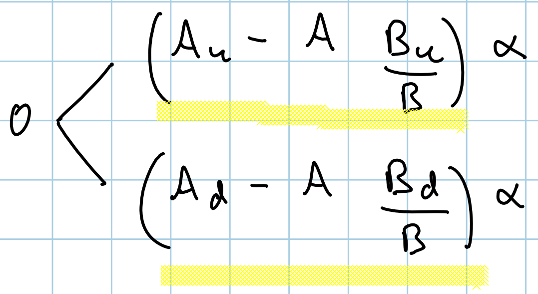


$B, B_u, B_d > 0$

$\frac{A_u}{B_u} > \frac{A_d}{B_d}$

$\alpha A + \beta B = 0$ ⇒ $\beta = -\alpha \frac{A}{B}$

$(V_0 = 0)$



$$\left.\begin{array}{l} A_u - A\frac{B_u}{B} > 0 \\ \\ A_d - A\frac{B_d}{B} < 0 \end{array}\right\} \Rightarrow \boxed{\begin{array}{l} \frac{B_u}{B} < \frac{A_u}{A} \\ \frac{A_d}{A} < \frac{B_d}{B} \end{array}}$$

no arb range

$B_u = B_d = (1+r)B$

no arb range

$$\frac{A}{B} < \frac{A_u}{B_u}$$

$$\frac{A}{B} > \frac{A_d}{B_d}$$

$$B_u = B_d = (1+r)B$$

$$\boxed{\frac{A_d}{A} < 1+r < \frac{A_u}{A}}$$

$$\boxed{\frac{A_d}{B_d} < \frac{A}{B} < \frac{A_u}{B_u}}$$

$$\begin{tikzcd}
 & \left(\frac{A_u}{B_u}\right) \\
\left(\frac{A}{B}\right) \arrow[ur, no head] \arrow[dr, no head] & \\
 & \left(\frac{A_d}{B_d}\right)
\end{tikzcd}$$

$$\boxed{\exists \text{ no arbitrage iff } \frac{A_d}{B_d} < \frac{A}{B} < \frac{A_u}{B_u}}$$

Three Assets

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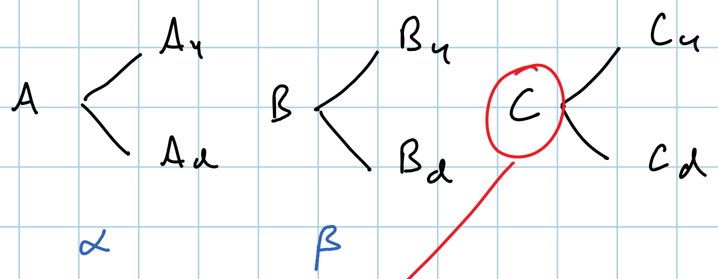


$$\alpha A+\beta B \begin{cases} \alpha A_u + \beta B_u = C_u & \text{— ①} \\ \alpha A_d + \beta B_d = C_d & \text{— ②} \end{cases}$$

$$\Rightarrow \quad \alpha = \dots \quad , \quad \beta = \dots$$

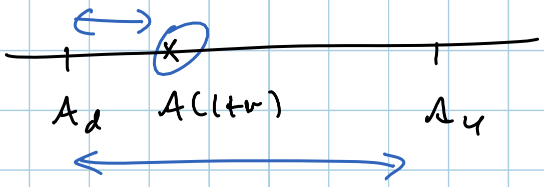
no arb iff $\quad C = \alpha A + \beta B$

· $B_u = B_d = (1+r) B$

$$\Rightarrow \quad C = \frac{1}{1+r} \underbrace{\left(q \; C_u + (1-q) \; C_d \right)}_{\hookrightarrow \; \mathbb{E}^{Q}[C_1]}$$

$$q = \frac{(1+r)A - A_d}{A_u - A_d}$$

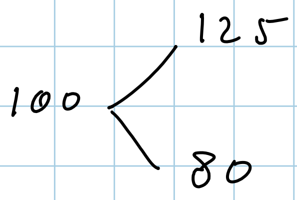
no arb iff $\quad A_d < A(1+r) < A_u$

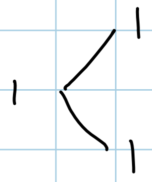


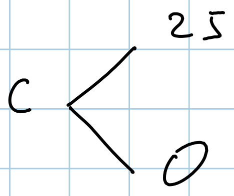
$$\boxed{\begin{array}{l} \therefore \ \exists \ \mathbb{Q} \ \text{s.t.} \quad C = \frac{1}{1+r} \mathbb{E}^{\mathbb{Q}}[C_1] \\ \text{iff} \ \nexists \ \text{arb.} \end{array}}$$

$\max(A_T - K ; 0)$

call on A struck @ 100







$$\left.\begin{array}{l} 125\alpha + \beta = 25 \\ 80\alpha + \beta = 0 \end{array}\right] \qquad 45\alpha = 25$$

$$\Rightarrow \alpha = \frac{25}{45} = \frac{5}{9}$$

$$\beta = -80 \times \frac{5}{9}$$

$$C = \frac{5}{9} 100 - 80 \times \frac{5}{9} \times 1 = \frac{5}{9}(20) \checkmark$$

$$100 = \frac{1}{1+0}\left(q\,125 + (1-q)\,80 \right) = 80 + 45q$$

$$\Rightarrow q = \frac{20}{45} = \frac{4}{9}$$

$$C = \frac{1}{1+0}\left(\frac{4}{9} \cdot 25 + \frac{5}{9} \cdot 0 \right) = \frac{4}{9} \cdot 25 \checkmark$$

Numeraire

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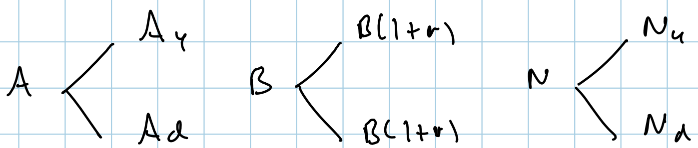
N is a numeraire asset ($N_\cdot > 0$ a.s.)

$$\boxed{\exists\, Q^N \text{ s.t. } \frac{C_0}{N_0} = \mathbb{E}^{Q^N}\left[\frac{C_1}{N_1}\right] \quad \text{iff } \nexists \text{ arb.}}$$

Q^N is the probability measure corresponding to the numeraire N.



$$\exists\, Q \text{ s.t. } \quad A_0 = \frac{1}{1+r}\mathbb{E}^{Q}[A_1] \quad (*)$$

$$N_0 = \frac{1}{1+r}\mathbb{E}^{Q}[N_1]$$

want: $$\frac{A_0}{N_0} = \underbrace{q^N}_{\in (0,1)} \frac{A_u}{N_u} + (1-q^N)\frac{A_d}{N_d}$$

$$(*) \Leftrightarrow \frac{A_0}{B_0} = q\frac{A_u}{B_u} + (1-q)\frac{A_d}{B_d}$$

$$\Leftrightarrow \frac{A_0}{N_0} = \underbrace{\left(\frac{B_0}{N_0}\, q\, \frac{N_u}{B_u}\right)}_{\tilde{q}} \frac{A_u}{N_u} + \underbrace{\frac{B_0}{N_0}(1-q)\left(\frac{N_d}{B_d}\right)}_{= 1-\tilde{q}\ ?} \frac{A_d}{N_d}$$

$$\tilde{q} \in (0,1)\ ?$$

$$\frac{B_0}{N_0}\left(q\,\frac{N_u}{B_u} + (1-q)\,\frac{N_d}{B_d}\right)$$

$$= \frac{\cancel{B_0}}{N_0}\left(\frac{\mathbb{E}^{\mathbb{Q}}[N_1]}{(1+r)\cancel{B_0}}\right) \hookrightarrow N_0 \quad = 1$$

$$\begin{tikzcd} \mathbb{P} \arrow[r] & \mathbb{Q} \arrow[r] & \mathbb{Q}^N \end{tikzcd}$$

$$C_0 = \mathbb{E}^{\mathbb{Q}}\left[\frac{C_1}{1+r}\right]$$

special case

$$\frac{C_0}{N_0} = \mathbb{E}^{\mathbb{Q}^N}\left[\frac{C_1}{N_1}\right]$$

$$\tilde{C}_t = \frac{C_t}{N_t}$$

$$\boxed{\tilde{C}_0 = \mathbb{E}^{\mathbb{Q}^N}[\tilde{C}_1]}$$

$\exists$ $\mathbb{Q}^N$ s.t. relatives prices of traded assets are $\mathbb{Q}^N$-martingales $\iff$ no arb.

Default Model

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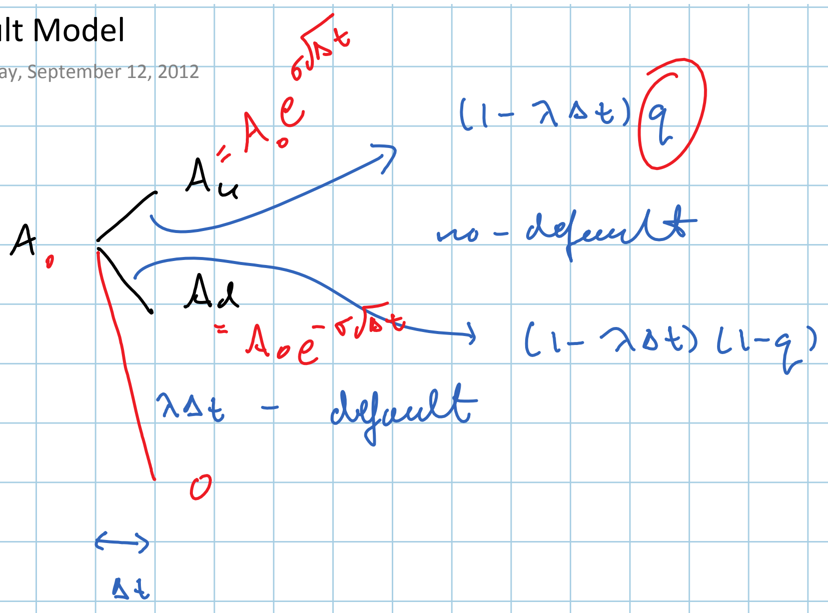
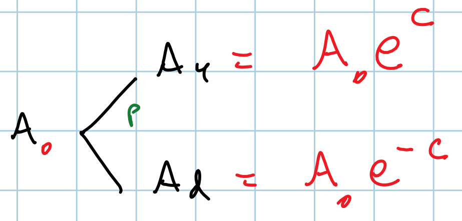
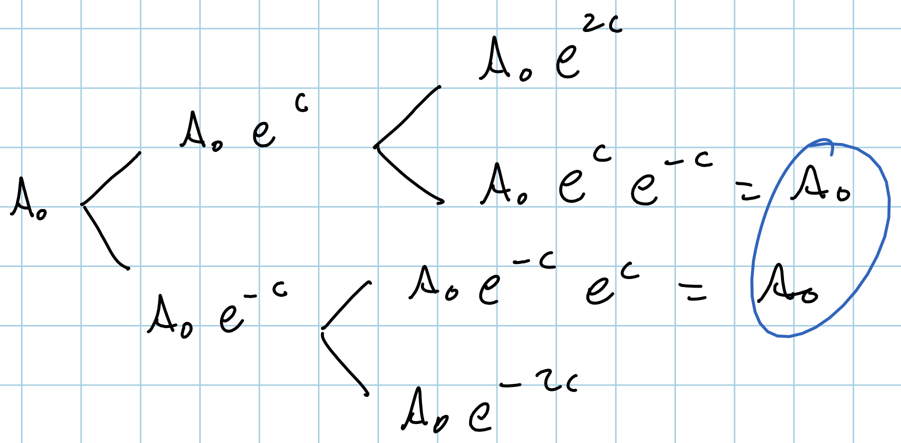






$$r_n = \ln\left(\frac{A_{n\Delta t}}{A_{(n-1)\Delta t}}\right)$$

$$\mathbb{E}[r] = \mu\Delta t$$

$$\mathbb{V}[r] = \sigma^2\Delta t$$

$$A_{n\Delta t} = A_{(n-1)\Delta t}\, e^{c\, x_n}$$

↳ Bernoulli r.v.

$\mathbb{P}(x_n = +1) = p$

$\mathbb{P}(x_n = -1) = 1-p$

$$\mathbb{E}[r] = c\,p + (-c)(1-p) = (2p-1)\,c = \mu\,\Delta t$$

$$\mathbb{V}[r] = c^2 - (\mu\,\Delta t)^2 = \sigma^2\,\Delta t$$

to order $(\Delta t)^{1/2}$ ... $\quad c = \sigma\sqrt{\Delta t}$

$$\Rightarrow \quad 2p-1 = \frac{\mu\,\Delta t}{c} = \frac{\mu}{\sigma}\sqrt{\Delta t}$$

$$\Rightarrow \quad p = \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{\Delta t}\right)$$

$$\begin{tikzcd}
 & A_0\,e^{\sigma\sqrt{\Delta t}} \\
A_0 \arrow[ur, dash] \arrow[dr, dash] & \\
 & A_0\,e^{-\sigma\sqrt{\Delta t}}
\end{tikzcd}$$

$$\frac{A_0}{B_0} = q\,\frac{A_0\,e^{\sigma\sqrt{\Delta t}}}{B_0\,e^{r\Delta t}} + (1-q)\,\frac{A_0\,e^{-\sigma\sqrt{\Delta t}}}{B_0\,e^{r\Delta t}}$$

↖ 1+r

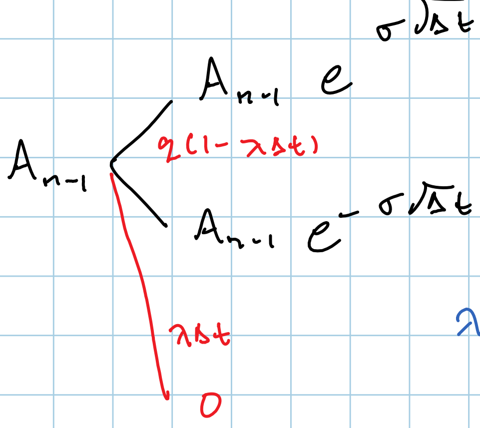
$$\Rightarrow \quad \boxed{q = \frac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}}$$

$$= \frac{1}{2}\left(1 + \frac{r - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right) + \cdots$$

$$p = \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{\Delta t}\right)$$

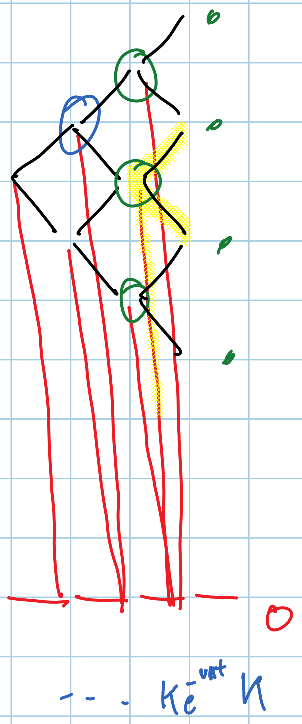
$$\mathbb{E}^{\mathbb{Q}}[r] = (r - \tfrac{1}{2}\sigma^2)\Delta t$$

$$\mathbb{V}^{\mathbb{Q}}[r] = \sigma^2 \Delta t$$

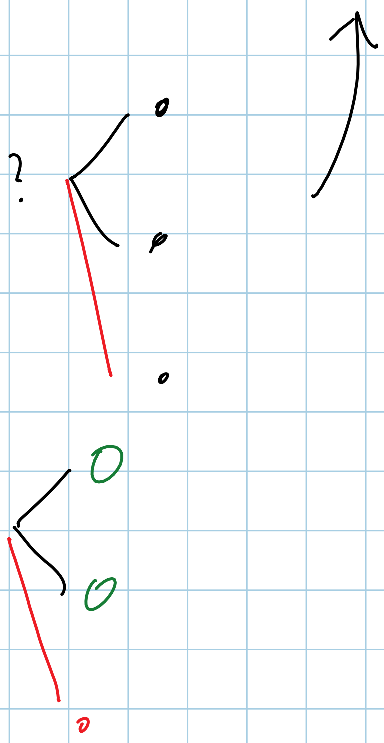


$\Rightarrow \quad q = ?$

λ- fixed & known.



$$C_{n-1} = e^{-r\Delta t}\,\mathbb{E}^{\mathbb{Q}}[C_n]$$



hold exer

hold in Q-

rΔt

$$\max\left(\underset{\uparrow}{C^{hold}_{n-1}} , \underset{\uparrow}{C^{exer}_{n-1}} \right) = ?$$

$$C^{hold}_{n-1} = \mathbb{E}^{Q}\left[C_n e^{-r\Delta t} \right]$$

$$C^{exer}_{n-1} = (K - A_{n-1})_+$$

↳ $(\cdot)_+ = \max(\cdot, 0)$

American protected put pays 0 upon default.