Black-Scholes PDE

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- Ito Calculus
- Feynman - Kac
- self-financing strategies

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$$A \begin{cases} A_u \\ A_d \end{cases} \qquad B \begin{cases} B_1 \\ B_1 \end{cases} \qquad C \begin{cases} C_u = \alpha A_u + \beta B_1 \\ C_d = \alpha A_d + \beta B_1 \end{cases}$$

$\alpha \qquad \beta$

$$C = \alpha A + \beta B$$
$$= \frac{1}{1+r} \mathbb{E}^{\mathbb{Q}}[C_1]$$

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① market driven by a single B.mtn W_t $(\mathbb{P})$

drift

$$dX_t = \mu_X(t, X_t)\,dt + \sigma_X(t, X_t)\,dW_t$$

mkt index

volatility

GBM: $\mu_X = \underline{\mu} X_t, \qquad \sigma_X = \underline{\sigma} X_t$

② traded asset, whose price $f_t = f(t, X_t)$ and

$f \in C^{1,2}$ and so we write:

$$\underline{df_t} = \mu^f(t, X_t)\,dt + \sigma^f(t, X_t)\,dW_t$$

f_t

(3) traded Bank acct whose price B_t and satisfies

$$dB_t = r(t, X_t)\, B_t\, dt$$

$$\Leftrightarrow \quad B_t = \exp\left\{ \int_0^t r(u, X_u)\, du \right\}$$

short rate of interest.

e.g. $r(t, x) = x$

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(4) ¿ What is value of a ctg. claim on X_t paying $\varphi(X_T)$ at $t = T$?

call this value $g_t = g(t, X_t)$ (also valid for American, Barrier)

$\in C^{1,2}$

$$\frac{dg_t}{g_t} = \mu^g(t, X_t)\, dt + \sigma^g(t, X_t)\, dW_t$$

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take a portfolio (α_t, β_t) in (f_t, B_t) & -1 in g_t.

$$V_t = \alpha_t f_t + \beta_t B_t - g_t$$

i) make $V_0 = 0$ by choosing α_0, β_0

ii) $V_{t+dt} - V_t = dV_t$

$$dV_t = d(\alpha_t f_t) + d(\beta_t B_t) - dg_t$$

$$= d\alpha_t f_t + \alpha_t df_t + d\beta_t B_t + \beta_t dB_t + d[\alpha, f]_t + d[\beta, B]_t - dg_t$$

$\hookrightarrow 0$

$\hookrightarrow 0$

self-financing constraint.

$$dV_t = \alpha_t df_t + \beta_t dB_t - dg_t$$

$$= \alpha_t f_t (\mu_t^f dt + \sigma_t^f dW_t) + \beta_t B_t r_t dt - g_t (\mu_t^g dt + \sigma_t^g dW_t)$$

iii) $\Rightarrow dV_t = (\alpha_t f_t \mu_t^f + \beta_t B_t r_t - g_t \mu_t^g) dt + (\alpha_t f_t \sigma_t^f - g_t \sigma_t^g) dW_t$

$\hookrightarrow 0$

$$\Rightarrow \quad \boxed{\alpha_t = \frac{g_t}{f_t} \frac{\sigma_t^g}{\sigma_t^f}} \qquad \text{need } \sigma_t^f > 0$$

iv) $$dV_t = \left[\mu_t^f \frac{\sigma_t^g}{\sigma_t^f} g_t + \beta_t B_t r_t - g_t \mu_t^g \right] dt$$

$\hookrightarrow$ 0 to avoid arbitrage.

v) $V_0 = 0$, $dV_t = 0$ $\Rightarrow$ $V_t = 0 = \alpha_t f_t + \beta_t B_t - g_t$

$$\Rightarrow \quad \boxed{\beta_t = \frac{g_t - g_t \sigma_t^g / \sigma_t^f}{B_t}}$$

vi) $$\Rightarrow \quad 0 = g_t \left[\mu_t^f \frac{\sigma_t^g}{\sigma_t^f} - \mu_t^g \right] + r_t \left(g_t - g_t \frac{\sigma_t^g}{\sigma_t^f} \right)$$

$$\Rightarrow \quad \boxed{\frac{\mu_t^f - r_t}{\sigma_t^f} = \frac{\mu_t^g - r_t}{\sigma_t^g}}$$

$\longrightarrow$ not specific to the asset g or f.
it is a function of the market

Sharpe Ratio (market price of risk)

$$\boxed{\frac{\mu_t^g - r_t}{\sigma_t^g} = \lambda_t = \lambda(t, X_t)}$$

$$dg_t = (\partial_t + \mathcal{L})\, g(t, X_t)\, dt + \partial_x g(t, X_t)\, \sigma^X(t, X_t)\, dW_t$$

$$\mathcal{L} h(t,x) = \mu^X(t,x)\, \partial_x h + \tfrac{1}{2}(\sigma^X(t,x))^2\, \partial_{xx} h$$

recall that,

$$dg_t = g_t\, \mu_t^g\, dt + g_t\, \sigma_t^g\, dW_t$$

$$\frac{\mu_t^g - r_t}{\sigma_t^g} = \lambda_t$$

$$\Leftrightarrow \quad \partial_t g_t + (\mu_t^X - \lambda_t\, \sigma_t^X)\, \partial_x g_t + \tfrac{1}{2}(\sigma_t^X)^2\, \partial_{xx} g_t = r_t\, g_t$$

$g(t, X_t)$

$$\partial_t g(t,x) + \left(\mu^X(t,x) - \lambda(t,x)\, \sigma^X(t,x) \right) \partial_x g(t,x) + \frac{1}{2} \left(\sigma_t^X(t,x)\right)^2 \partial_{xx} g(t,x) = r(t,x)\, g(t,x)$$

$\forall\, t \in [0,T)$, $x \in \mathbb{R}$

b.c. $g(T,x) = \varphi(x)$

Pricing PDE
(generalized B-S PDE)

Black-Scholes Model

- $dX_t = \mu X_t\, dt + \sigma X_t\, dW_t$
- $f(t,x) = x$
- $r(t,x) = r$ const.

$$\lambda_t = \frac{\mu_t^g - r_t}{\sigma_t^g} \quad \text{choose } g \text{ to be } X_t;$$

$$= \frac{\mu - r}{\sigma}$$

$$\boxed{\partial_t g + \underbrace{\left(\mu x - \left(\frac{\mu - r}{\sigma}\right) \cdot \sigma x \right)}_{r x} \partial_x g + \frac{1}{2} \sigma^2 x^2 \partial_{xx} g = r g}$$

$g(T,x) = \varphi(x)$

Feynman-Kac Theorem

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Feynman-Kac Thm:

suppose that a fn $h \in C^{1,2}$ satisfies:

$$\begin{cases} \partial_t h(t,x) + a(t,x)\,\partial_x h(t,x) + \frac{1}{2}\sigma^2(t,x)\,\partial_{xx} h(t,x) = c(t,x)\,h(t,x) \\ h(T,x) = H(x) \end{cases}$$

Then it admits a representation:

$$h(t,x) = \mathbb{E}^{\mathbb{A}}\left[H(X_T)\, e^{-\int_t^T c(u, X_u)\,du} \,\middle|\, X_t = x \right]$$

where $dX_t = a(t, X_t)\,dt + \sigma(t, X_t)\,dW_t^{\mathbb{A}}$

$\mathbb{A}$ - B. mtn.

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sketch of Proof:

recall: $\eta_t \triangleq \mathbb{E}[\,G \mid \mathcal{F}_t\,]$ $\quad G \in \mathcal{F}_T$

is an $\mathbb{A}$-mtg (Doob mtg)

$$\left(\mathbb{E}[\,\eta_s \mid \mathcal{F}_t\,] = \mathbb{E}\big[\, \mathbb{E}[\,G \mid \mathcal{F}_s\,] \mid \mathcal{F}_t \,\big] \right.$$

$(T > s > t > 0)$

$$= \mathbb{E}[\,G \mid \mathcal{F}_t\,]$$

$$= \eta_t \qquad)$$

$$e^{-\int_0^t c(u,X_u)\,du}\ \mathbb{E}[\overset{\zeta_t}{\cdots} \mid \mathcal{F}_t]$$

$$= \mathbb{E}[\underbrace{H(X_T)\, e^{-\int_0^T c(u,X_u)\,du}}_{\in \mathcal{F}_T} \mid \mathcal{F}_t] \triangleq \eta_t \quad \text{is a mtg}$$

$$\mathbb{E}[\eta_{t+\Delta T} - \eta_t \mid \mathcal{F}_t] = 0 \qquad \forall\, \Delta T > 0.$$

$$d\eta_t = d(\zeta_t\, e^{-\int_0^t c_u\,du})$$

$$= d\zeta_t\, e^{-\int_0^t c_u\,du} + \zeta_t(-c_u)\, e^{-\int_0^t c_u\,du}\,dt$$

$$+ \underbrace{d[\zeta, e^{-\int_0^\cdot c_u\,du}]_t}_{0}$$

Ito's lemma

$$d\zeta_t = (\partial_t h + a(t,X_t)\partial_x h + \tfrac{1}{2}b^2(t,X_t)\partial_{xx} h)\,dt$$

$$+ b(t,X_t)\partial_x h\, dW_t$$

$$\Rightarrow \boxed{[\partial_t + a(t,X_t)\partial_x + \tfrac{1}{2}b^2(t,X_t)\partial_{xx} - c(t,X_t)]\, h(t,X_t) = 0}$$

replace $X_t \to x$ since must hold $\forall$ paths.

$$\zeta_T = \mathbb{E}\left[H(X_T)\, e^{-\int_T^T c_u\,du} \,\middle|\, \mathcal{F}_T \right]$$

$$= H(X_T)$$

$$h(T, x) = H(x)$$

The Risk-Neutral Measure

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So since asset price g_t satisfies,

$$\begin{cases} \partial_t g + (\mu^x - \lambda \sigma^x)\, \partial_x g + \frac{1}{2}(\sigma^x)^2\, \partial_{xx} g = r\, g \\ \qquad\qquad\qquad\qquad g(T, x) = \varphi(x) \end{cases}$$

we can write: (F-K)

$$g(t,x) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_u\, du}\, \varphi(X_T) \,\middle|\, X_t = x \right]$$

where

$$dX_t = \left(\mu^x(t, X_t) - \lambda(t, X_t)\, \sigma^x(t, X_t)\right) dt + \sigma^x(t, X_t)\, dW_t^{\mathbb{Q}}$$

$\llcorner$ $\mathbb{Q}$- B. mtn.

$$= \mu^x(t, X_t)\, dt + \sigma^x(t, X_t)\, dW_t$$

$\llcorner$ $\mathbb{P}$- B. mtn.

$$dW_t = dW_t^{\mathbb{Q}} - \lambda(t, X_t)\, dt$$

$$\frac{dg_t}{g_t} = \mu^g_t\, dt + \sigma^g_t\, dW_t \qquad\qquad \frac{\mu^g_t - r_t}{\sigma^g_t} = \lambda_t$$

g_t $\qquad$ t

$$= (r_t + \lambda_t \sigma_t^g)\,dt + \sigma_t^g\,dW_t$$

$$= r_t\,dt + \sigma_t^g\,dW_t^{\mathbb{Q}}$$

So g_t has drift of r_t under $\mathbb{Q}$-prob.