Black-Scholes Formula

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$$(K - S_\tau)_+$$

$$\tau = T$$

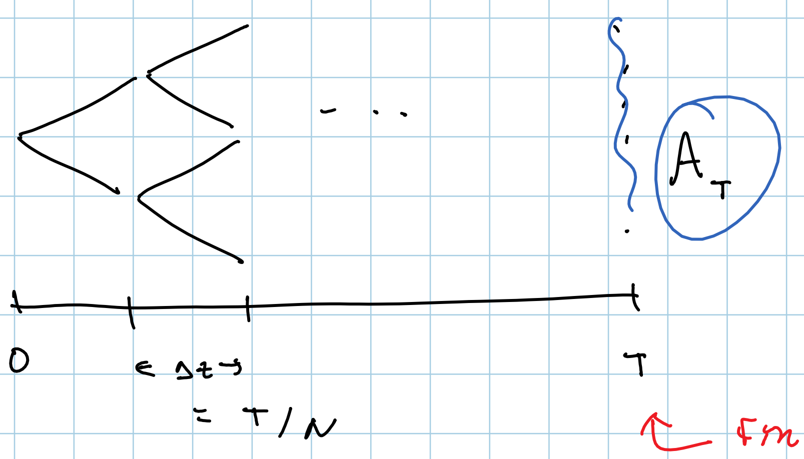
$$(K - S_\tau)_+ \, \mathbb{1}_{S_\tau > 0}$$

$$A_n = A_{n-1} \, e^{\sigma \sqrt{\Delta t} \, x_n}$$

$x_1, x_2, \ldots$ iid Bernoulli r.v.

$$\mathbb{P}(x_1 = +1) = p = \frac{1}{2}\left(1 + \frac{\mu - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right) + \cdots$$

$$\mathbb{P}(x_1 = -1) = 1 - p$$



$N \to +\infty$.

$$A_T = A_0 \exp\{\underbrace{\sigma\sqrt{\Delta t}\,(x_1 + x_2 + \cdots + x_N)}_{X}\}$$

$$X \xrightarrow[N \uparrow +\infty]{\mathcal{L}, \mathbb{P}} \mathcal{N}\left((\mu - \tfrac{1}{2}\sigma^2)T, \; \sigma^2 T\right)$$

CLT

$$\mathbb{E}^{\mathbb{P}}[X] = (\mu - \tfrac{1}{2}\sigma^2)\, T$$

$$\left(\mathbb{E}[A_T] = A_0 e^{\mu T} \right)$$

$$\mathbb{V}^{\mathbb{P}}[X] = \sigma^2 T$$

$$\lim_{N \uparrow +\infty} \quad \boxed{A_T \overset{d}{=} A_0 \, e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\, Z}, \quad Z \underset{\mathbb{P}}{\sim} \mathcal{N}(0,1)}$$

"Black-Scholes model"

Asset price is lognormally distributed.

risk-neutral probability measure $\mathbb{Q}$ has

$$q = \mathbb{Q}(x_1 = +1) = \frac{1}{2}\left(1 + \frac{r - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right) + \cdots$$

$$\mathbb{E}^{\mathbb{Q}}[\ln(A_T/A_0)] = (r - \tfrac{1}{2}\sigma^2)\,T$$

$$\mathbb{V}^{\mathbb{Q}}[\ln(A_T/A_0)] = \sigma^2 T$$

$$X \xrightarrow[N \uparrow +\infty]{d,\,\mathbb{Q}} \mathcal{N}\left((r - \tfrac{1}{2}\sigma^2)T \,;\, \sigma^2 T\right)$$

by CLT

$$\lim_{N \uparrow +\infty} \quad \boxed{A_T \overset{d}{=} A_0 \, e^{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\, Z}, \quad Z \underset{\mathbb{Q}}{\sim} \mathcal{N}(0,1)}$$

Value of a call struck @ K mat is T.

$$(\max(A_T - K, 0) = (A_T - K)_+)$$

$$\frac{V_0}{B_0} = \mathbb{E}^{\mathbb{Q}}\left[\frac{V_T}{B_T} \right]$$

$B_0 \hookleftarrow 1$, $\quad B_T \hookrightarrow e^{rT}$

$$\Rightarrow V_0 = e^{-rT}\, \mathbb{E}^{\mathbb{Q}}[V_T]$$

$$= e^{-rT}\, \mathbb{E}^{\mathbb{Q}}[(A_T - K)_+]$$

$$= e^{-rT}\, \mathbb{E}^{\mathbb{Q}}\left[\left(A_0\, e^{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T} Z} - K\right)_+\right]$$

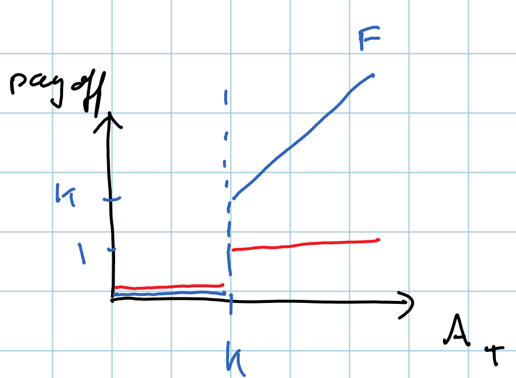
$$Z \underset{\mathbb{Q}}{\sim} \mathcal{N}(0,1)$$

$\longrightarrow$ can use explicit computation.

$$V_T = (A_T - K)_+ = \underbrace{A_T\, \mathbb{1}_{A_T > K}}_{\text{asset or nothing}} - K\, \underbrace{\mathbb{1}_{A_T > K}}_{\text{cash or nothing}}$$

asset or nothing: F

cash or nothing (digital option / binary): D

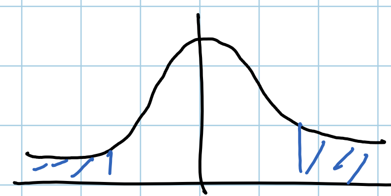


$$D_0 = e^{-rT}\,\mathbb{E}^{\mathbb{Q}}[D_T] = e^{-rT}\,\mathbb{E}^{\mathbb{Q}}[\mathbb{1}_{A_T > K}]$$

$$= e^{-rT}\,\mathbb{Q}(A_T > K)$$

$$= e^{-rT}\,\mathbb{Q}\left(A_0 e^{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T} Z} > K\right), \quad Z \underset{\mathbb{Q}}{\sim} N(0,1)$$

$$= e^{-rT}\,\mathbb{Q}\left(Z > \frac{1}{\sigma\sqrt{T}}\left(\ln(K/A_0) - (r - \tfrac{1}{2}\sigma^2)T\right)\right)$$



$$D_0 = e^{-rT}\,\Phi\Bigg(\underbrace{\frac{\ln(A_0/K) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}}_{d_-}\Bigg)$$

$$F_0 = e^{-rT}\,\mathbb{E}^{\mathbb{Q}}[A_T \mathbb{1}_{A_T > K}]$$

$$\frac{F_0}{A_0} = \mathbb{E}^{\mathbb{Q}^A}\left[\frac{F_T}{A_T}\right] = \mathbb{E}^{\mathbb{Q}^A}[\mathbb{1}_{A_T > K}]$$

$$\Rightarrow \quad F_0 = A_0\,\mathbb{Q}^A(A_T > K)$$

$$A \left\langle \begin{matrix} A e^{\sigma\sqrt{\Delta t}} \\ A e^{-\sigma\sqrt{\Delta t}} \end{matrix} \right. \qquad 1 \left\langle \begin{matrix} e^{r\Delta t} \\ e^{r\Delta t} \end{matrix} \right.$$

$$\downarrow$$

$$r\Delta t - \sigma\sqrt{\Delta t}$$

A^{-1}

$$A^{-1} \begin{cases} q^A & e^{r\Delta t + \sigma\sqrt{\Delta t}} \quad A^{-1} \\ \end{cases}$$

$1 + r\Delta t + \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2 \Delta t$

$$q^A = \frac{1 - e^{r\Delta t + \sigma\sqrt{\Delta t}}}{e^{r\Delta t - \sigma\sqrt{\Delta t}} - e^{r\Delta t + \sigma\sqrt{\Delta t}}} \quad \begin{matrix} \sim -\left((r + \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}\right) \\ \sim -2\sigma\sqrt{\Delta t} \end{matrix}$$

$1 + r\Delta t - \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2 \Delta t \qquad 1 + r\Delta t + \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2 \Delta t$

$$q^A \sim \frac{1}{2}\left[1 + \frac{r + \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right] + \cdots$$

$$p \sim \frac{1}{2}\left[1 + \frac{\mu - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right] + \cdots$$

$$q \sim \frac{1}{2}\left[1 + \frac{r - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right] + \cdots$$

$\mu \to r + \sigma^2$

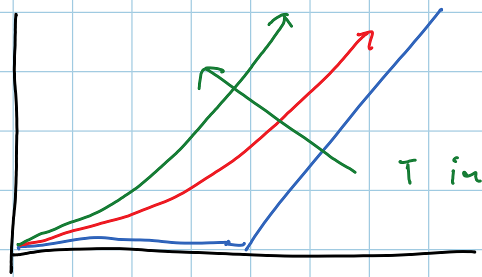
$\mu \to r$

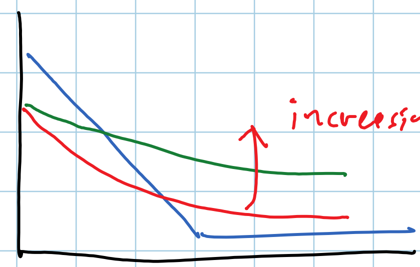
$r \to r + \sigma^2$

$$\boxed{\lim_{N \to +\infty} A_T \overset{d}{=} A_0 \exp\left\{ (r + \tfrac{1}{2}\sigma^2) T + \sigma\sqrt{T}\, Z \right\} \qquad Z \underset{\mathbb{Q}^A}{\sim} N(0,1)}$$

so then, $$\mathbb{Q}^A(A_T > K) = \Phi\Bigg(\underbrace{\frac{\ln(A_0/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}}_{d_+} \Bigg)$$

$$\boxed{V_0 = A_0\,\Phi(d_+) - K e^{-rT}\,\Phi(d_-)}$$





Incomplete Markets

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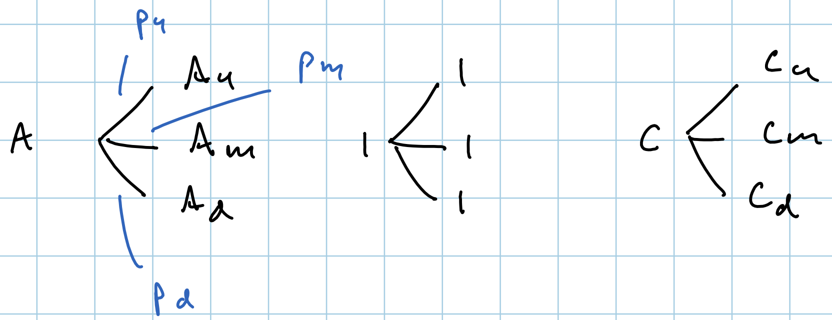


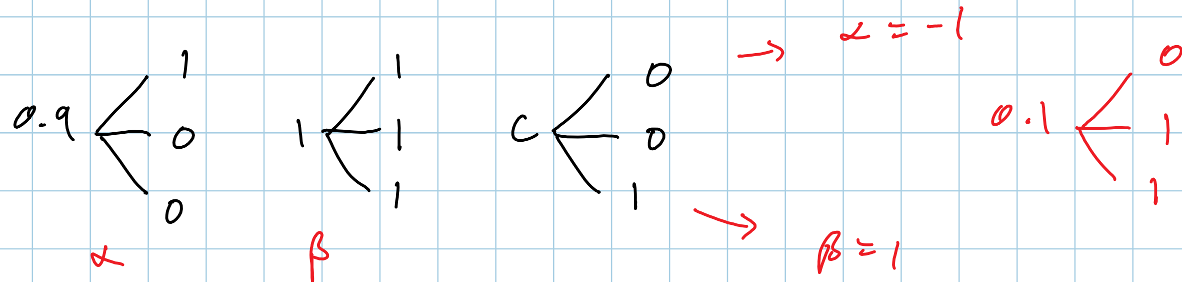
$$A = q_u A_u + q_m A_m + q_d A_d$$

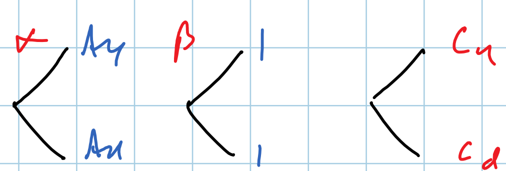
$$1 = q_u + q_m + q_d$$

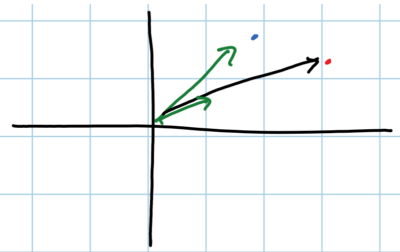
$\mathbb{Q}$ is not necessarily unique!

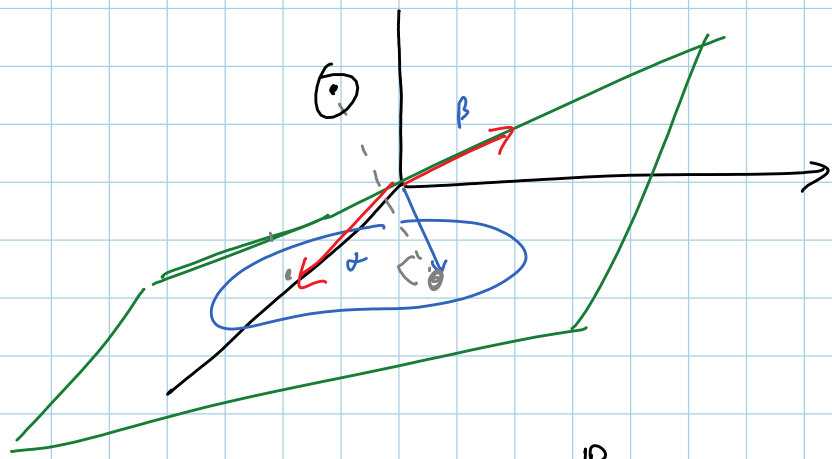
$$C = q_u C_u + q_m C_m + q_d C_d$$ but not unique

range of no arbitrage pricing!









$$(\alpha^*, \beta^*) = \underset{(\alpha,\beta)}{\arg\min}\ \mathbb{E}^{\mathbb{P}}\left[\left(\alpha A_1 + \beta - C_1 \right)^2 \right]$$

$$C_0 = \alpha^* A_0 + \beta^*$$

minimum variance hedge / replication

$$\partial_\alpha (\ ) = 0 \Rightarrow \mathbb{E}^{\mathbb{P}}[(\alpha A_1 + \beta - C_1) A_1] = 0 \quad ①$$

$$\partial_\beta (\ ) = 0 \Rightarrow \mathbb{E}^{\mathbb{P}}[\alpha A_1 + \beta - C_1] = 0 \quad ②$$

$$\hookrightarrow \mathbb{E}(C_1) = \alpha \mathbb{E}[A_1] + \beta \quad ③$$

$$① \Rightarrow \alpha \mathbb{E}[A_1^2] + \beta \mathbb{E}[A_1] = \mathbb{E}[C_1 A_1] \quad ④$$

$$④ - \mathbb{E}[A_1]③$$

$$\Rightarrow \alpha\left(\mathbb{E}[A_1^2] - (\mathbb{E}[A_1])^2 \right) = \mathbb{E}[C_1 A_1] - \mathbb{E}[C_1]\mathbb{E}[A_1]$$

$$\Rightarrow \alpha = \mathbb{C}[A_1, C_1]$$

$$\Rightarrow \quad \alpha = \frac{\mathbb{C}[A_1, C_1]}{\mathbb{V}[A_1]}$$

$$\Rightarrow \quad \beta = \mathbb{E}[C_1] - \frac{\mathbb{C}[A_1, C_1]}{\mathbb{V}[A_1]} \cdot \mathbb{E}[A_1]$$

then

$$\begin{aligned} C_0 &= \alpha A_0 + \beta \\ &= \mathbb{E}[C_1] + \left(A_0 - \mathbb{E}[A_1]\right) \frac{\mathbb{C}[A_1, C_1]}{\mathbb{V}[A_1]} \end{aligned}$$

Interest Rate Tree

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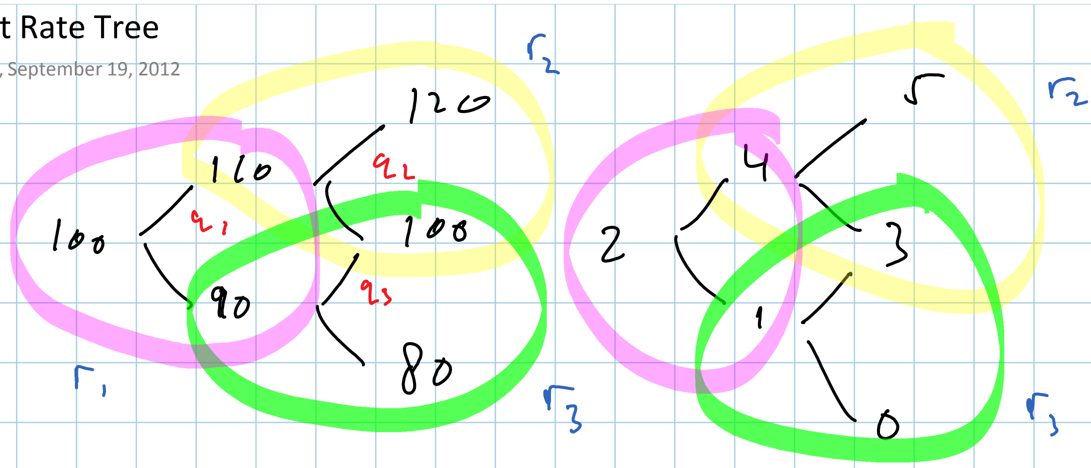
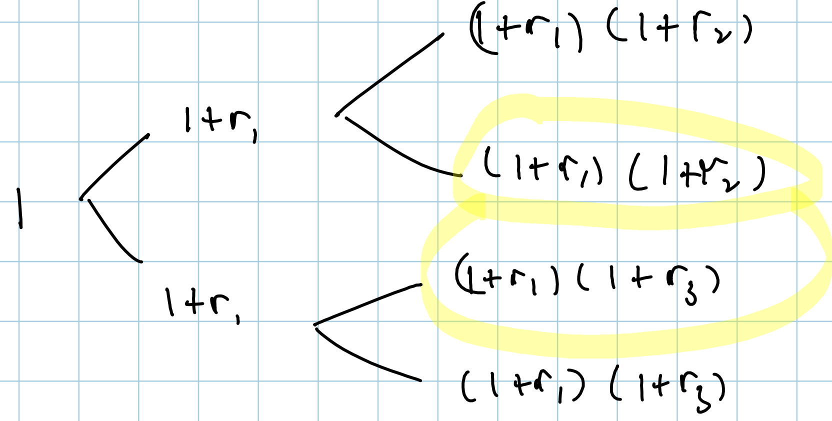
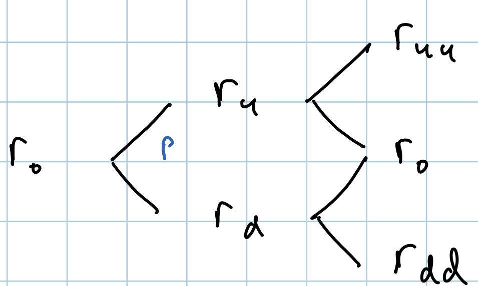


$$110 = \frac{1}{1+r_1}\left(120\, q_2 + 100\,(1-q_2)\right)$$

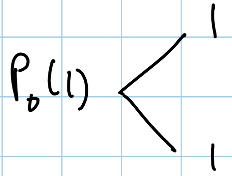
$$4 = \frac{1}{1+r_2}\left(5\, q_2 + 3\,(1-q_2)\right)$$



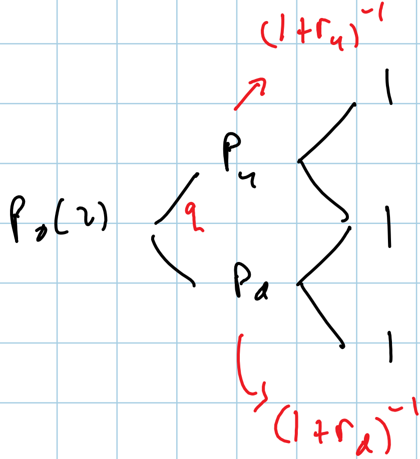
Interest Rate Tree



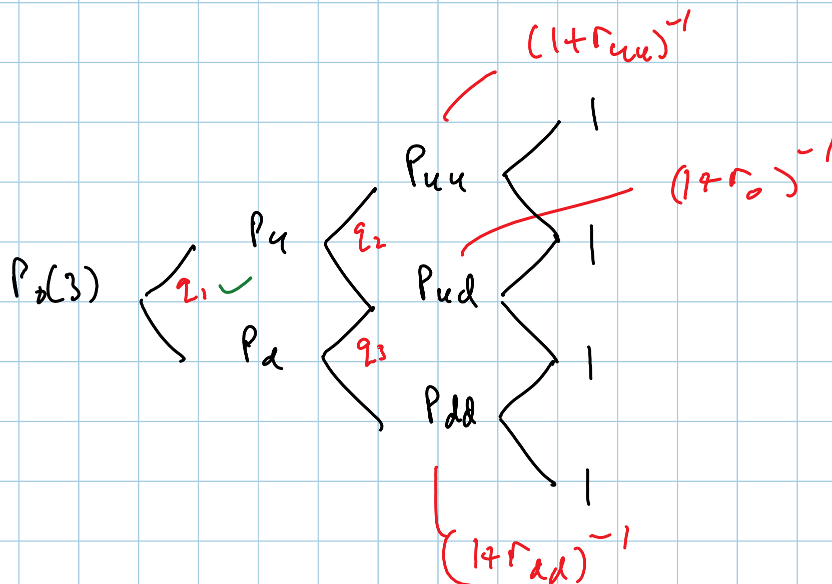
look at zero coupon bonds:



$$\frac{P_0(1)}{1} = \mathbb{E}^{Q}\left[\frac{P_1}{1+r_0}\right] = \frac{1}{1+r_0}$$



$$P_0(2) = \frac{1}{1+r_0}\left(q\,(1+r_u)^{-1} + (1-q)\,(1+r_d)^{-1}\right)$$



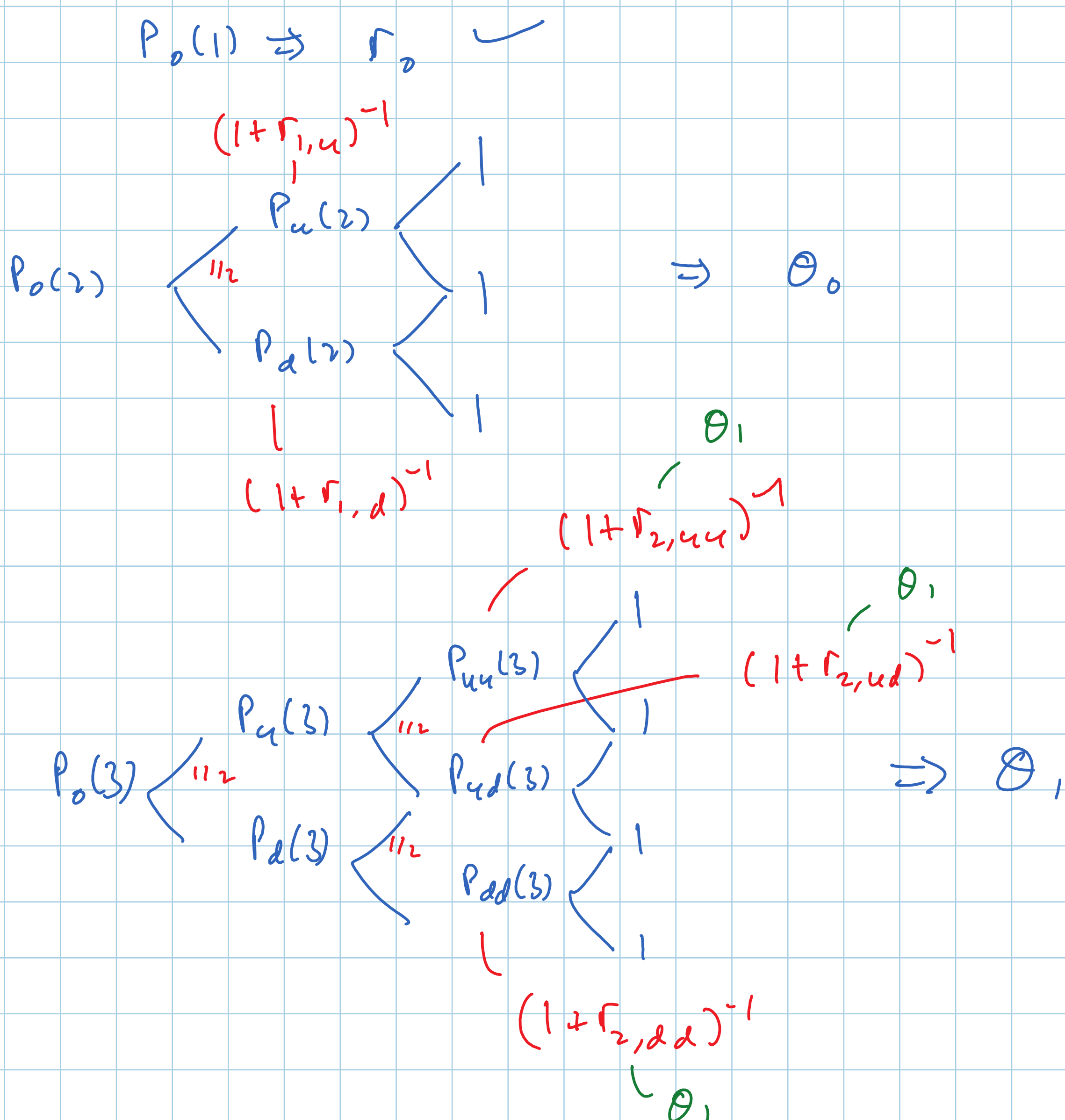
Pick Q then find IR Tree consistent with bond prices.

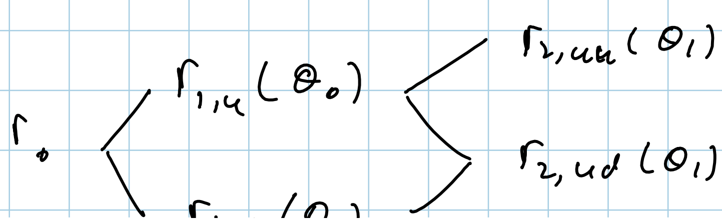
$$r_n = r_{n-1} + \theta_{n-1}\Delta t + \sigma_{n-1}\sqrt{\Delta t}\, x_n$$

$x_1, x_2, \ldots$ iid Bernoulli r.v.

$\mathbb{Q}(x_1 = \pm 1) = \frac{1}{2}$.

goal find θ's s.t. Bonds are matched exactly. & r_0





r_0

$r_{1,d}(\theta_0)$

$r_{2,ud}(\theta_1)$

$r_{2,dd}(\theta_1)$

Arrow-Debreu Securities

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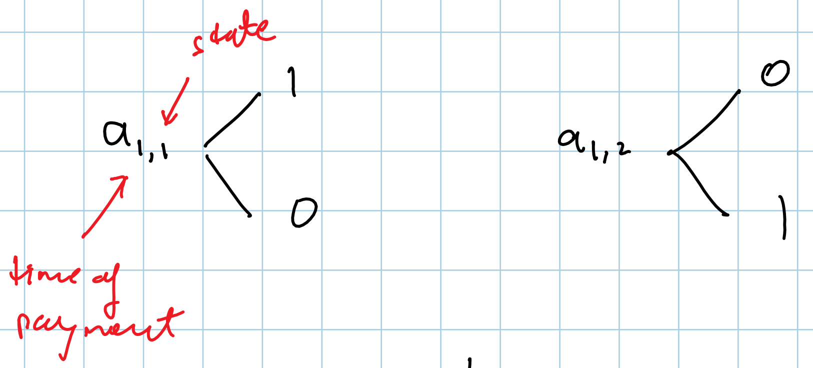
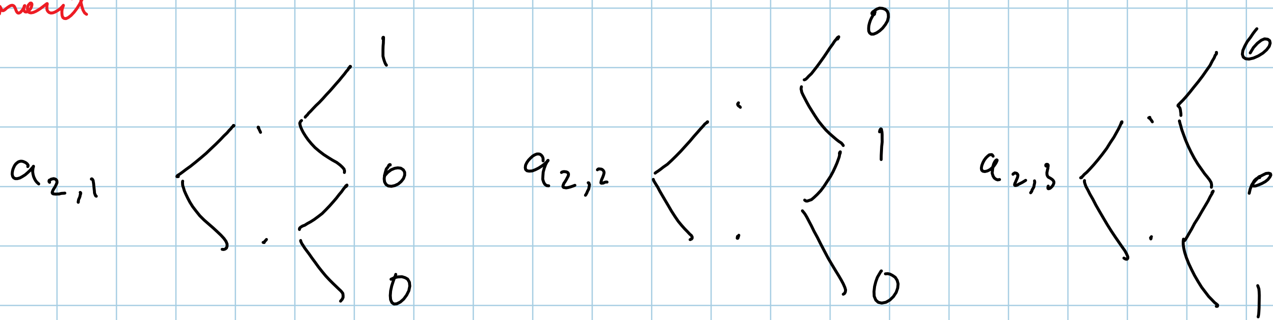
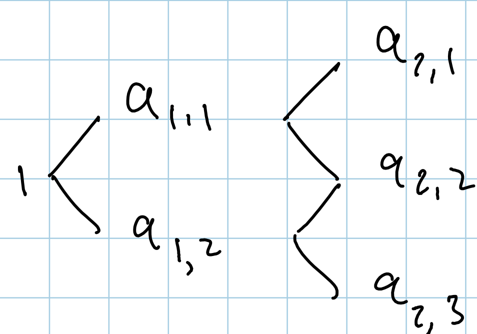
Arrow-Debreu Securities

are assets that pay 1 at t if state m prevails, and pay 0 otherwise.







time is fixed at $t=0$

$S_t \triangleq$ state prevailing at time t

$$a_{n,m}(t=n) = \mathbb{1}_{S_n = m}$$

want $a_{n,m}(0)$

$$a_{n,m}(0) = \mathbb{E}^{\mathbb{Q}}\left[(d_0 \cdot d_1 \cdot d_2 \cdots d_{n-1})\, \mathbb{1}_{S_n = m} \right]$$

$e^{-r_0 \Delta t}$ $\left(\frac{1}{1+r_0}\right)$

$$= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[d_0 \cdots d_{n-1}\, \mathbb{1}_{S_n = m} \mid S_{n-1} \right] \right]$$

$$= \mathbb{E}^{\mathbb{Q}}\Big[\underbrace{\mathbb{E}^{\mathbb{Q}}\left[d_0 \cdots d_{n-2}\, \mathbb{1}_{S_{n-1,m-1}} \right]}_{a_{n-1,m-1}(0)} q^d_{n-1,m-1}\, d_{n-1,m-1}$$

$$+ \cdots \Big]$$

$d_{n-1,m}$, $r_{n-1,m}$, $S_{n-1} = m$