Wednesday, October 27, 2010
10:05 AM

Black-Scholes:

- $$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t$$ ⌐ $\mathbb{P}$-Brownian motion. traded

- $$\frac{dM_t}{M_t} = r\, dt$$ traded

- value a claim g_t pays $\varphi(S_T)$ @ T.

$$\frac{\mu^g - r}{\sigma^g} = \lambda(t, S_t) = \frac{\mu^S - r}{\sigma^S} = \frac{\mu - r}{\sigma} \longleftarrow$$

$$\frac{dg_t}{g_t} = \mu^g_t\, dt + \sigma^g_t\, dW_t$$

Ito's lemma

$$= \frac{1}{g_t}\Big[\big(\partial_t g + \mu S_t \partial_s g + \tfrac{1}{2}\sigma^2 S_t^2 \partial_{ss} g\big)\, dt + \sigma S_t \cdot \partial_s g\, dW_t\Big]$$

$$\Rightarrow \quad \mu^g g - r g = \left(\frac{\mu - r}{\sigma}\right) \sigma^g g$$

$$\Rightarrow \quad \big(\partial_t g + \underline{\mu S \partial_s g} + \tfrac{1}{2}\sigma^2 S^2 \partial_{ss} g\big) - r g = \underline{\left(\frac{\mu - r}{\sigma}\right) \sigma S \partial_s g}$$

$$\boxed{\begin{array}{l} \partial_t g + r S \partial_s g + \tfrac{1}{2}\sigma^2 S^2 \partial_{ss} g = r g \\ \qquad\qquad\qquad g(T, S) = \varphi(S) \end{array}}$$

$$\begin{cases} \partial_t g + r S \partial_S g + \frac{1}{2}\sigma^2 S^2 \partial_{SS} g = r g \\ \qquad\qquad g(T,S) = \Phi(S) \end{cases}$$

$$g(t,S) = e^{-r(T-t)} h(t,S)$$

$$\partial_t g = r g + e^{-r(T-t)} \partial_t h$$

$$\partial_S g = e^{-r(T-t)} \partial_S h \,, \quad \partial_{SS} g = e^{-r(T-t)} \partial_{SS} h$$

$$\Rightarrow \begin{cases} \partial_t h + r S \partial_S h + \frac{1}{2}\sigma^2 S^2 \partial_{SS} h = 0 \,, \\ \qquad\qquad h(T,S) = \Phi(S) \,. \end{cases}$$

$$x = \ln S \iff S = e^x$$

$$\frac{\partial}{\partial S} h = \frac{\partial x}{\partial S} \frac{\partial}{\partial x} h = \frac{1}{S} \partial_x h$$

$$\partial_{SS} h = \frac{\partial}{\partial S}\left(\frac{1}{S} \partial_x h \right) = -\frac{1}{S^2} \partial_x h + \frac{1}{S} \partial_S (\partial_x h)$$

$$= -\frac{1}{S^2} \partial_x h + \frac{1}{S}\left(\frac{1}{S} \partial_x (\partial_x h) \right)$$

$$= \frac{1}{S^2} \left(-\partial_x h + \partial_{xx} h \right)$$

$$\begin{cases} \partial_t h + (r - \frac{1}{2}\sigma^2) \partial_x h + \frac{1}{2}\sigma^2 \partial_{xx} h = 0 \\ \qquad\qquad h(T,x) = \Phi(e^x) \end{cases}$$

$$h(t,x) = \ell(t, f(t,x))$$

$$\partial_t h(t,x) = \partial_t \ell(t, f(t,x)) + \partial_x \ell(t, f(t,x)) \cdot \partial_t f(t,x)$$

$$\partial_x h(t,x) = \partial_x \ell(t, f(t,x)) \cdot \partial_x f(t,x)$$

$$\partial_{xx} h(t,x) = \partial_{xx} \ell(t, f(t,x)) \cdot \left(\partial_x f(t,x)\right)^2$$

$$+ \partial_x \ell(t, f(t,x)) \cdot \partial_{xx} f(t,x)$$

$$\text{PDE} \Rightarrow \quad \partial_t \ell + \partial_x \ell \; \underbrace{\left(\partial_t f + (r - \tfrac{1}{2}\sigma^2)\, \partial_x f \right)}_{\to 0}$$

$$+ \tfrac{1}{2}\sigma^2 \, \partial_{xx} \ell \; \underbrace{(\partial_x f)^2}_{\to 1}$$

$$+ \tfrac{1}{2}\sigma^2 \, \partial_x \ell \; \underbrace{\partial_{xx} f}_{\to 0} \qquad = 0$$

try $\quad f(t,x) = a\,t + \overset{1}{b}\,x$

$$\partial_t f + (r - \tfrac{1}{2}\sigma^2)\partial_x f = a + \underbrace{b}_{1}\,(r - \tfrac{1}{2}\sigma^2) = 0$$

$$\Rightarrow a = -(r - \tfrac{1}{2}\sigma^2)$$

$$f(t,x) = x + (r - \tfrac{1}{2}\sigma^2)(T - t)$$

$$\boxed{\begin{cases} \partial_t \ell + \tfrac{1}{2}\sigma^2 \, \partial_{xx} \ell = 0 \\ \ell(T,x) = \varphi(e^x) \end{cases}}$$

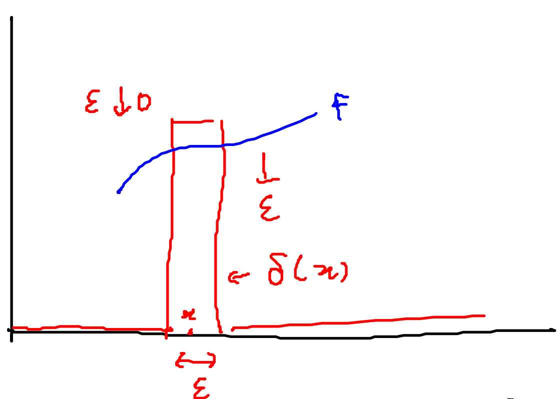
Backwards heat equation

$$g(t,S) = e^{-r(T-t)} \, h(t, \ln S)$$

$$= e^{-r(T-t)} \, \ell\left(t, \ln S + (r - \tfrac{1}{2}\sigma^2)(T-t)\right)$$

$$\begin{cases} \partial_t k + \frac{1}{2}\partial_{xx} k = 0 \\ k(T,x) = \varphi(e^{x\sigma}) \end{cases}$$

$$g(t,S) = e^{-r(T-t)}\, k\left(t, \frac{\ln S + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma}\right)$$



$$\lim_{\varepsilon \downarrow 0} \int_{x-\delta}^{x+\delta} f(u)\cdot \delta_\varepsilon(u-x)\, du$$

$$= f(x)$$

$$\ell(t,x) = \frac{e^{-\frac{1}{2}\frac{x^2}{\sigma^2(T-t)}}}{\sqrt{2\pi\sigma^2(T-t)}}$$ is our fundamental solution at 0.

Wednesday, October 27, 2010
11:31 AM

$$\begin{cases} \partial_t g + r S \partial_S g + \frac{1}{2}\sigma^2 S^2 \partial_{SS} g = r g \\ g(T,S) = \varphi(S) \end{cases}$$

$(\Omega, (\mathcal{F}_t)_{0 \le t \le T}, \mathbb{P})$ $\hookrightarrow$ dynamic hedging, no arb $\Uparrow$

Feynman-Kac Thm:

$$f(t, S_t) = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}\left[\varphi(S_T) \mid \mathcal{F}_t \right]$$

$$\frac{dS_t}{S_t} = r\,dt + \sigma\, d\widehat{W}_t$$

$\llcorner$ $\mathbb{Q}$- Brownian mtn

then,
$$\begin{cases} \partial_t f + r S \partial_S f + \frac{1}{2}\sigma^2 S^2 \partial_{SS} f = r f \\ f(T,S) = \varphi(S) . \end{cases}$$

consider $h_t = e^{r(T-t)} f(t, S_t) = \mathbb{E}^{\mathbb{Q}}\left[\varphi(S_T) \mid \mathcal{F}_t \right]$

is in fact a martingale ...

$$\underset{(u<s)}{h_u} \overset{?}{=} \mathbb{E}^{\mathbb{Q}}\left[h_s \mid \mathcal{F}_u \right] = \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[\varphi(S_T) \mid \mathcal{F}_s \right] \mid \mathcal{F}_u \right]$$
$$= \mathbb{E}^{\mathbb{Q}}\left[\varphi(S_T) \mid \mathcal{F}_u \right] = h_u$$

so ..

$$\mathbb{E}^{\mathbb{Q}}\left[h_{t+\Delta t} - h_t \mid \mathcal{F}_t \right] = 0$$

but $$h_{t+\Delta t} - h_t = \int_t^{t+\Delta t} (\partial_t + \mathcal{L})\, h_u\, du + \int_t^{t+\Delta t} \sigma S_u\, \partial_S h_u\, d\widehat{W}_u$$

$$\mathcal{L} = r S \partial_S + \frac{1}{2} \sigma^2 S^2 \partial_{SS}$$

$$\Rightarrow \quad \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{\Delta t} \int_t^{t+\Delta t} (\partial_t + \mathcal{L})\, h_u\, du \;\Big|\; \mathcal{F}_t \right] = 0$$

$$\lim_{\Delta t \downarrow 0} \mathbb{E}^{\mathbb{Q}}_t\left[\frac{1}{\Delta t} \int_t^{t+\Delta t} (\partial_t + \mathcal{L})\, h_u\, du \right]$$

$$= \mathbb{E}^{\mathbb{Q}}_t\left[\lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} \int_t^{t+\Delta t} (\partial_t + \mathcal{L})\, h_u\, du \right]$$

$$= \mathbb{E}^{\mathbb{Q}}_t\left[(\partial_t + \mathcal{L})\, h_t \right] = (\partial_t + \mathcal{L})\, h_t = 0$$

Wednesday, October 27, 2010
11:58 AM

$$\alpha_t = \frac{g_t \, \sigma_t^g}{S_t \, \sigma}$$ is the hedge for B-S.

$$= \partial_S g$$ is called the Delta of the option

$$g(t,S) = e^{-r(T-t)} \, \mathbb{E}^{\mathbb{Q}}\left[\varphi(S_T) \mid S_t = S \right]$$

$$\frac{dS_t}{S_t} = r\,dt + \sigma\, d\widehat{W}_t$$

$$g(t,S) = e^{-r(T-t)} \, \mathbb{E}^{\mathbb{Q}}\left[\varphi(S e^X) \right]$$

$$X \overset{d}{=} (r - \tfrac{1}{2}\sigma^2)(T-t) + \sigma\sqrt{T-t}\; Z$$

$$Z \underset{\mathbb{Q}}{\sim} \mathcal{N}(0,1)$$

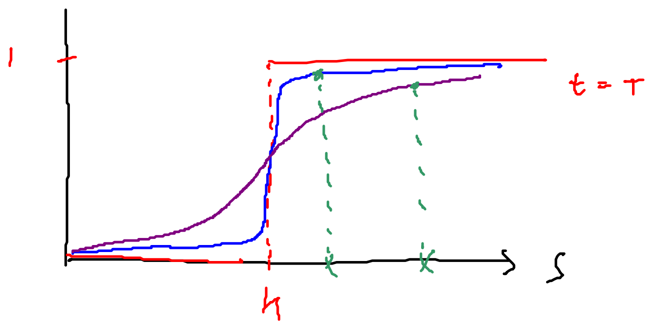
so $$\Delta_t = \partial_S g$$
$$= e^{-r(T-t)} \, \mathbb{E}^{\mathbb{Q}}\left[\varphi'(S e^X)\, e^X \right]$$

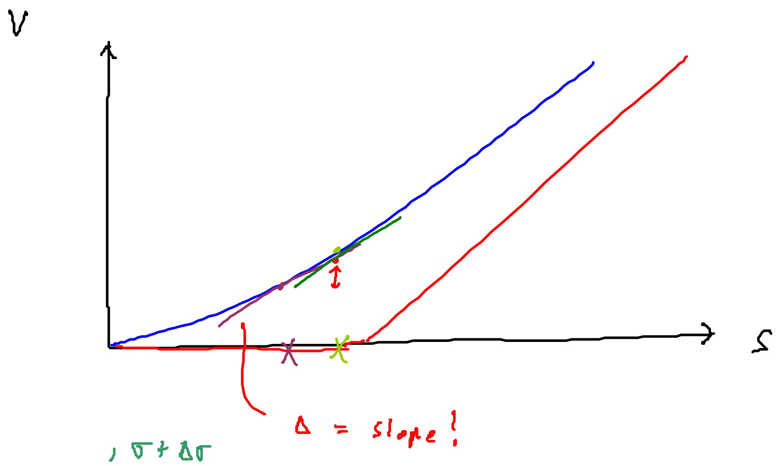
$$\varphi(S) = (S-K)_+ \,, \quad \varphi'(S) = \begin{cases} 0 & S < K \\ 1, & S \geq K \end{cases}$$

$$\Delta_t^{call} = e^{-r(T-t)} \, \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{S e^X \geq K} \; e^X \right]$$

$$= \cdots = \Phi(d_+)$$

$$d_+ = \frac{\ln(S/K) + (r + \tfrac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$





$$g(t+\Delta t,\ S+\Delta S,\ \sigma+\Delta\sigma) - g(t, S, \sigma)$$

$$\Delta g \sim \underbrace{\partial_S g}_{\substack{\Delta^g \\ \text{delta}}} (\Delta S) + \frac{1}{2} \underbrace{\partial_{SS} g}_{\substack{\Gamma^g \\ \text{Gamma}}} (\Delta S)^2 + \underbrace{\partial_\sigma g}_{\substack{\nu^g \\ \text{vega}}} (\Delta \sigma)$$

---

sell an option g

Delta hedge at equal pts in time...
(time-based)

$\underline{t=0}$: get g_0

hold Δ_0^g - units of S

$M_0 = (g_0 - \Delta_0^g S_0)$ left over

$\underline{t=\Delta t}$:
$$\left.\begin{array}{l} M_0 \longrightarrow M_0 e^{r\Delta t} \\ \Delta_0^g \longrightarrow \Delta_0^g \end{array}\right\} \quad M_0 e^{r\Delta t} + \Delta_0^g S_1 = \text{value now}$$

Delta has now changed! Δ_1^g

buy $(\Delta_1^g - \Delta_0^g)$ units of S

$$M_1 = M_0 e^{r\Delta t} - (\Delta_1^g - \Delta_0^g) S_1$$

repeat...

$$M_n = M_{n-1} e^{r\Delta t} - (\Delta_n^g - \Delta_{n-1}^g) S_n$$

Δ_n^g units of S_n

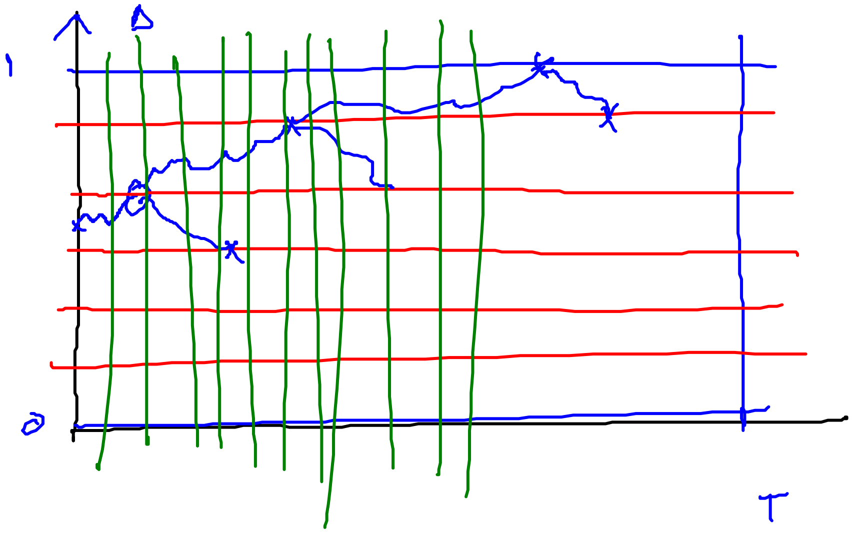
...

$t = T = N\Delta t$

M_N & Δ_N^g of S owe $Q(S_T)$

$$V_N = M_N + \Delta_N^g S_N - Q(S_T)$$

Wednesday, October 27, 2010
12:43 PM



transaction costs: $|\Delta_n^g - \Delta_{n-1}^g| S_n \cdot \alpha$

$\llcorner \frac{10}{100}$ %

Wednesday, October 27, 2010
12:58 PM

$\underline{t=0}$ sell an option g_0

α_0 units of S
β_0 units of h

$$\left.\begin{aligned} \alpha_0 1 + \beta_0 \Delta_0^h &= \Delta_0^g \\ \alpha_0 \cdot 0 + \beta_0 \Gamma_0^h &= \Gamma_0^g \end{aligned}\right\} \Rightarrow \alpha_0, \beta_0$$

$$M_0 = g_0 - (\alpha_0 S_0 + \beta_0 h_0)$$

$\underline{t=\Delta t}$

$$\left.\begin{aligned} M_0 &\to e^{r\Delta t} M_0 \\ \alpha_0 &\to \alpha_0 \\ \beta_0 &\to \beta_0 \end{aligned}\right\} \text{value} = e^{r\Delta t} M_0 + \alpha_0 S_1 + \beta_0 h_1$$

$\llcorner \tau = T - \Delta t$!

$$\begin{cases} \alpha_1 \cdot 1 + \beta_1 \cdot \Delta_1^h = \Delta_1^g \\ \beta_1 \Gamma_1^h = \Gamma_1^g \end{cases}$$

$$M_n = M_{n-1} e^{r\Delta t} - (\alpha_n - \alpha_{n-1}) S_n - (\beta_n - \beta_{n-1}) h_n$$