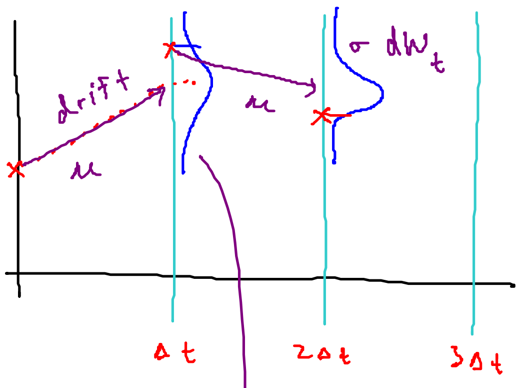
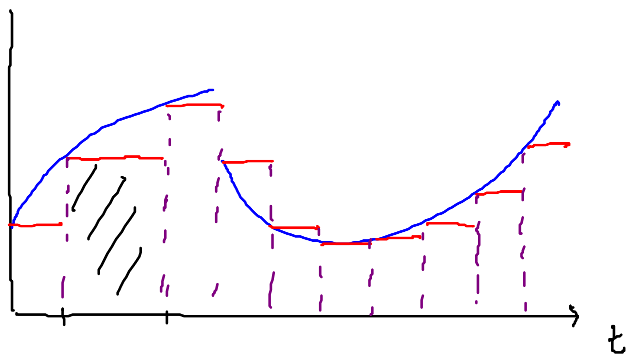


$$\sigma \underbrace{\left(W_{t+\Delta t} - W_t\right)}_{\text{"}dW_t\text{"}} \overset{d}{=} \sigma \sqrt{\Delta t}\, Z$$

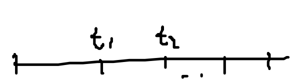


$$\int_0^t W_s \, dW_s \overset{?}{=} \frac{1}{2}\left(W_t^2 - t\right)$$

$$\int_0^t g_s \, dg_s = \frac{1}{2}\left(g_t^2 - g_0^2\right)$$

$$[W_t, W_t] = t \qquad a.s.$$

$$\| \quad \lim_{\|\Pi\| \downarrow 0} \sum_k \left(W_{t_k} - W_{t_{k-1}}\right)^2 = t \qquad a.s.$$



$0 = t_0 \qquad t = t_n$

$$A = \sum_k (W_{t_k} - W_{t_{k-1}})^2 - t$$

$$= \sum_k \underbrace{\left[(W_{t_k} - W_{t_{k-1}})^2 - (t_k - t_{k-1}) \right]}_{A_k}$$

$$\mathbb{E}[A_k] = \mathbb{E}[(W_{t_k} - W_{t_{k-1}})^2] - \Delta t_k$$

$$= \mathbb{V}[\Delta W_{t_k}] - \Delta t_k$$

$$= \Delta t_k - \Delta t_k = 0 \quad \ldots \quad (1)$$

$$\mathbb{V}[A_k] = \mathbb{V}[(\Delta W_{t_k})^2]$$

$$= \mathbb{E}[(\Delta W_{t_k})^4] - \left(\mathbb{E}[(\Delta W_{t_k})^2]\right)^2$$

$$= 3\,\Delta t_k^2 - \Delta t_k^2 = 2\,\Delta t_k^2 \xrightarrow[\|\pi\| \downarrow 0]{} 0$$

$$\mathbb{V}[A] = \sum_k \mathbb{V}[A_k] = 2 \sum_k \underbrace{\Delta t_k^2}_{\Delta t_k \, \Delta t_k}$$

$$\leq 2 \sum_k \Delta t_k \|\pi\| = 2 \|\pi\| t \xrightarrow[\|\pi\| \downarrow 0]{} 0 \quad \ldots \quad (2)$$

$$(1) + (2) \Rightarrow \sum_k (W_{t_k} - W_{t_{k-1}})^2 \xrightarrow[\|\pi\| \downarrow 0]{} t \quad a.s.$$

$$\Delta W_{t_k} \overset{d}{=} \sqrt{\Delta t_k}\, Z, \quad Z \sim \mathcal{N}(0,1)$$

$$\mathbb{E}[\Delta W_{t_k}^4] = \Delta t_k^2 \, \mathbb{E}[Z^4]$$

$$\mathbb{E}[Z^n] = \partial_a^n \underbrace{\mathbb{E}[e^{aZ}]}_{e^{\frac{1}{2}a^2}} \Big|_{a=0}$$

$$\partial_a e^{\frac{1}{2}a^2} = a \, e^{\frac{1}{2}a^2}$$

$$\partial_a^2 (\ ) = (1 + a^2) e^{\frac{1}{2}a^2}$$

$$\partial_a^3 (\ ) = (2a + a + a^3) e^{\frac{1}{2}a^2}$$
$$= (3a + a^3) e^{\frac{1}{2}a^2}$$

$$\partial_a^4 (\ ) = (3 + 3a^2 + 3a^2 + a^4) e^{\frac{1}{2}a^2}$$
$$= (3 + 6a^2 + a^4) e^{1/2 a^2}$$

$$\xrightarrow[a \to 0]{} 3$$

take a partition π ...

$$A = \sum_k W_{t_{k-1}} (W_{t_k} - W_{t_{k-1}}) - \frac{1}{2}(W_t^2 - t)$$

$$= \sum_k \left(W_{t_{k-1}} (W_{t_k} - W_{t_{k-1}}) - \frac{1}{2}(W_{t_k}^2 - W_{t_{k-1}}^2) + \frac{1}{2} \Delta t_k \right)$$

$\hookrightarrow -\frac{1}{2}\left((\Delta W_{t_k})^2 - \Delta t_k\right)$

$$\xrightarrow[\|\pi\| \downarrow 0]{} 0 \quad a.s.$$

$$\int_0^t W_s \, dW_s = \frac{1}{2}(W_t^2 - t) \quad a.s.$$

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$$\mathbb{E}[W_t] = 0$$

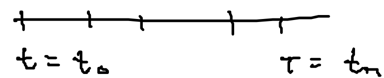
$$\mathbb{E}[W_t \mid \mathcal{F}_s] = \mathbb{E}[(W_t - W_s) + W_s \mid \mathcal{F}_s]$$

$$s < t \qquad = \mathbb{E}[W_s \mid \mathcal{F}_s]$$

$$= W_s$$

$$0 \overset{?}{=} \mathbb{E}\left[\int_t^T g_s \, dW_s \,\middle|\, \mathcal{F}_t\right]$$

$$\mathbb{E}\left[\sum_k g_{t_{k-1}} \Delta W_{t_k} \,\middle|\, \mathcal{F}_t\right]$$



$$= \sum_k \mathbb{E}[g_{t_{k-1}} \Delta W_{t_k} \mid \mathcal{F}_t]$$

$$\hookrightarrow \mathbb{E}[\mathbb{E}[g_{t_{k-1}} \Delta W_{t_k} \mid \mathcal{F}_{t_{k-1}}] \mid \mathcal{F}_t]$$

$$= \mathbb{E}[g_{t_{k-1}} \underbrace{\mathbb{E}[\Delta W_{t_k} \mid \mathcal{F}_{t_{k-1}}]}_{\hookrightarrow 0} \mid \mathcal{F}_t]$$

$$= 0$$

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convexity correction

$$X_t = e^{-\frac{1}{2}c^2 t + cW_t}$$

$$A_t \overset{d}{=} A_0 \, e^{(r - \frac{1}{2}\sigma^2)t + \sigma\sqrt{t}\, Z} \qquad Z \sim N(0,1)$$

$$\mathbb{E}[A_t] = A_0 e^{rt}$$

$$\mathbb{E}[X_T \mid \mathcal{F}_t] = \mathbb{E}[e^{-\frac{1}{2}c^2 T + cW_T} \mid \mathcal{F}_t]$$

$$= e^{-\frac{1}{2}c^2 T}\, \mathbb{E}[e^{c(W_T - W_t) + cW_t} \mid \mathcal{F}_t]$$

$$= e^{-\frac{1}{2}c^2 T + cW_t}\, \mathbb{E}[e^{c(W_T - W_t)} \mid \mathcal{F}_t]$$

$$\left(W_T - W_t \Big|_{\mathcal{F}_t} \overset{d}{=} \sqrt{T-t}\, Z \quad , Z \sim N(0,1) \right)$$

$$= e^{-\frac{1}{2}c^2 T + cW_t}\, e^{\frac{1}{2}(T-t)c^2}$$

$$= e^{-\frac{1}{2}c^2 t + cW_t} = X_t .$$

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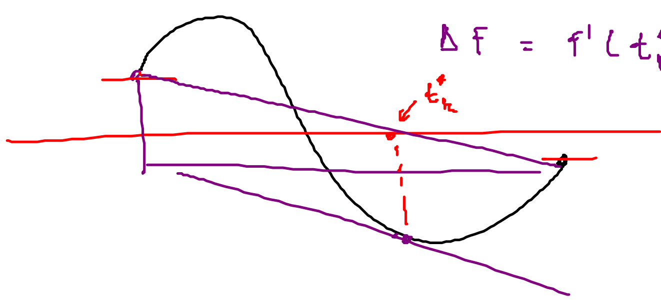
$$W_t \longmapsto f(W_t) = X_t$$

$\hookrightarrow$ 2-diff $\in C^2$

std. calc. $\quad dX_t = f'(W_t)\, dW_t$ $\quad$ ¿ how about when W_t is a B. mtn?

$f(W_t) - f(W_0)$ $\quad$ take partition π ...

$$= \sum_k \underbrace{\left(f(W_{t_k}) - f(W_{t_{k-1}}) \right)}$$



$$f(W_{t_k}) = f(W_{t_{k-1}} + \Delta W_{t_k})$$

$$= f(W_{t_{k-1}}) + \Delta W_{t_k} f'(W_{t_{k-1}})$$

$$+ \frac{1}{2} (\Delta W_{t_k})^2 f''(W_{t_{k-1}}) + \cdots$$

$$f(W_t) - f(W_0) = \sum_k \left(f(W_{t_k}) - f(W_{t_{k-1}}) \right)$$

$$= \underbrace{\sum_k \Delta W_{t_k} f'(W_{t_{k-1}})} \longrightarrow \int_0^t f'(W_s)\, dW_s$$

$$+ \frac{1}{2} \underbrace{\sum_k (\Delta W_{t_k})^2 f''(W_{t_{k-1}})} + \cdots$$

$$\hookrightarrow \int_0^t f''(W_s)\, ds \quad a.s.$$

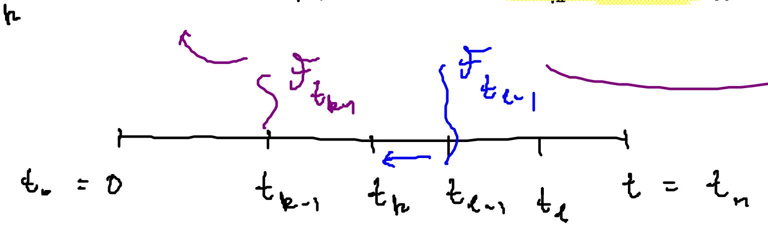
$$A = \sum_k \underbrace{\left(f''(W_{t_{k-1}}) \left((\Delta W_{t_k})^2 - \Delta t_k \right) \right)}_{\sim A_k}$$

$$\mathbb{E}[A_k] = \mathbb{E}\left[\mathbb{E}\left[f''(W_{t_{k-1}}) \left((\Delta W_{t_k})^2 - \Delta t_k \right) \mid \mathcal{F}_{t_{k-1}} \right] \right]$$

$$= 0$$

$$\mathbb{V}[A] = 2 \sum_{k<\ell} \mathbb{E}\Big[\left(f''(W_{t_{k-1}}) \left((\Delta W_{t_k})^2 - \Delta t_k \right) \right) \left(f''(W_{t_{\ell-1}}) \left((\Delta W_{t_\ell})^2 - \Delta t_\ell \right) \right) \Big]$$

$$+ \sum_k \mathbb{E}\left[\left(f''(W_{t_{k-1}}) \right)^2 \left((\Delta W_{t_k})^2 - \Delta t_k \right)^2 \right]$$



$$\mathbb{E}\left[\left((\Delta W_{t_k})^2 - \Delta t_k \right)^2 \mid \mathcal{F}_{t_{k-1}} \right]$$

$$= \mathbb{E}\left[(\Delta W_{t_k})^4 + (\Delta t_k)^2 - 2 (\Delta W_{t_k})^2 \Delta t_k \mid \mathcal{F}_{t_{k-1}} \right]$$

$$= (3 + 1 - 2) \Delta t_k^2 = 2 \Delta t_k^2$$

$$\mathbb{V}[A] = \sum_k \mathbb{E}\left[\left(f''(W_{t_{k-1}}) \right)^2 \right] 2 \underbrace{\Delta t_k^2}_{\leq \Delta t_k \|\pi\|}$$

$$\leq 2 \|\pi\| \sum_k \mathbb{E}\left[\left(f''(W_{t_{k-1}}) \right)^2 \right] \Delta t_k$$

$$= 2 \|\pi\| \, \mathbb{E}\left[\sum_k \left(f''(W_{t_{k-1}}) \right)^2 \Delta t_k \right]$$

$\hookrightarrow \mathbb{E}\left[\int_0^t \left(f''(W_s)\right)^2 ds\right]$ assume $< +\infty$.

$\xrightarrow[\|\Pi\| \downarrow 0]{} 0$

$$\boxed{f(W_t) - f(W_0) = \frac{1}{2}\int_0^t f''(W_s)\,ds + \int_0^t f'(W_s)\,dW_s}$$

$$df = \frac{1}{2} f''\,dt + f'\,dW_t$$

$f(t, W_t)$ add term: $\int_0^t \partial_1 f(s, W_s)\,ds$

OR $+\ \partial_1 f(t, W_t)\,dt$

$$dX_t = \mu(t, X_t)\,dt + \sigma(t, X_t)\,dW_t$$

$f(t, X_t)$

$$\boxed{\begin{aligned} df = &\left(\partial_t f + \mu\,\partial_x f + \frac{1}{2}\sigma^2\,\partial_{xx} f\right) dt \\ &+ \sigma\,\partial_x f\,dW_t \end{aligned}}$$

$$\begin{aligned} dt &\to dt \\ (\Delta W)^2 &\to dt \\ \Delta W\,dt &\to 0 \\ dt^2 &\to 0 \\ \Delta W &\to dW \end{aligned}$$

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$$\int_0^t W_s \, dW_s = \frac{1}{2}(W_t^2 - t)$$

$$f(x) = x^2$$

$$d f(W_t) = \frac{1}{2} f''(W_t)\, dt + f'(W_t)\, dW_t$$

$$= \frac{1}{2}\, 2\, dt + 2 \cdot W_t \cdot dW_t$$

$$\Rightarrow \quad W_t \, dW_t = \frac{1}{2} d(W_t^2) - \frac{1}{2} dt$$

$$\int_0^t W_s \, dW_s = \frac{1}{2} \int_0^t d(W_t^2) - \frac{1}{2} t$$

$$= \frac{1}{2}(W_t^2 - t)$$