value a call option on an asset A
strike $= K$, mat $= T$ under the CRR model $\xrightarrow[N \to \infty]{}$ log-normal

$$A_T \overset{d}{=} A_0 \exp\{ (\mu - \tfrac{1}{2}\sigma^2) T + \sigma\sqrt{T}\, Z \}$$

$$Z \underset{\mathbb{P}}{\sim} N(0,1)$$

$$A_T \overset{d}{=} A_0 \exp\{ (r - \tfrac{1}{2}\sigma^2) T + \sigma\sqrt{T}\, Z \}$$

$$Z \underset{\mathbb{Q}}{\sim} N(0,1)$$

$$V_0 = e^{-rT}\, \mathbb{E}^{\mathbb{Q}}\left[(A_T - K)_+ \right]$$

$$= e^{-rT}\, \mathbb{E}^{\mathbb{Q}}\left[\left(A_0\, e^{(r-\frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\, Z} - K\right)_+ \right]$$

$$= e^{-rT} \int_{-\infty}^{\infty} \left(A_0\, e^{(r-\frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\, z} - K\right)_+ \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}\, dz$$

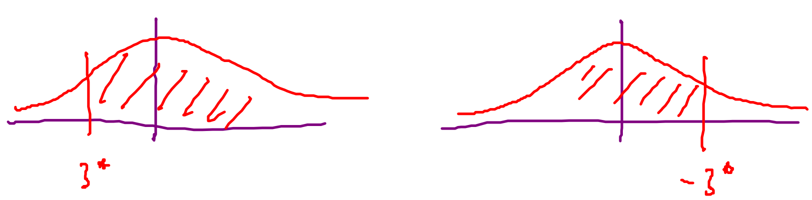
$$= e^{-rT} \int_{z^*}^{\infty} \left(A_0\, e^{(r-\frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\, z} - K\right) \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}\, dz$$

where $A_0\, e^{(r-\frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\, z^*} = K$

$$\Leftrightarrow \quad z^* = - \frac{\ln A_0/K + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$\int_{z^*}^{\infty} K\, \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}\, dz = K\, \Phi(-z^*)$$

$\hookrightarrow$ cdf of std normal



$$\int_{z_*}^{\infty} e^{\sigma\sqrt{T} z} \frac{e^{-\frac{1}{2} z^2}}{\sqrt{2\pi}}\, dz = \int_{z_*}^{\infty} e^{-\frac{1}{2}(z - \sigma\sqrt{T})^2 + \frac{1}{2}\sigma^2 T} \frac{dz}{\sqrt{2\pi}}$$

$$= \int_{z_* - \sigma\sqrt{T}}^{\infty} e^{-\frac{1}{2} z^2} \frac{dz}{\sqrt{2\pi}} \quad e^{+\frac{1}{2}\sigma^2 T}$$

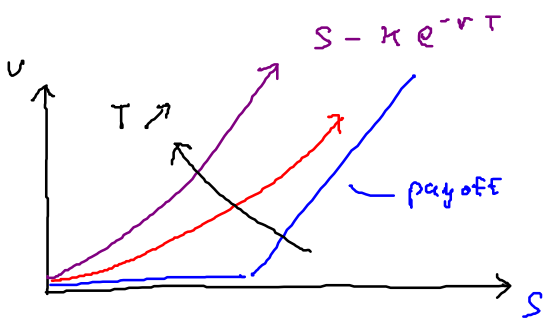
$$= e^{+\frac{1}{2}\sigma^2 T} \quad \Phi(-z_* + \sigma\sqrt{T})$$

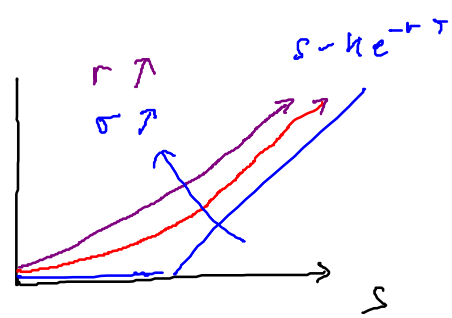
$$\boxed{\begin{aligned} V_0 &= A_0\, \Phi(d_+) - K e^{-rT}\, \Phi(d_-) \\ d_\pm &= \frac{\ln(A_0/K) + (r \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \end{aligned}}$$

Black-Scholes Formula for call option

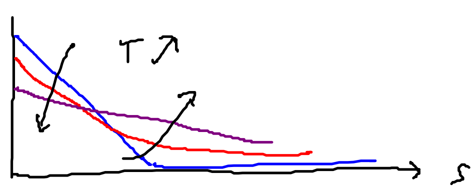
$$V_0 = K e^{-rT}\, \Phi(-d_-) - A_0\, \Phi(-d_+) \quad \leftarrow \text{ for a put}$$

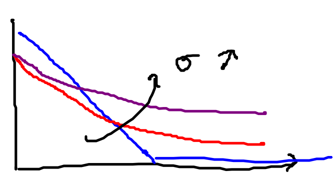
call

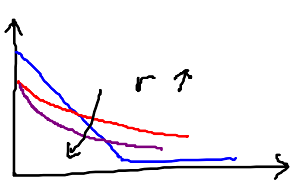




put

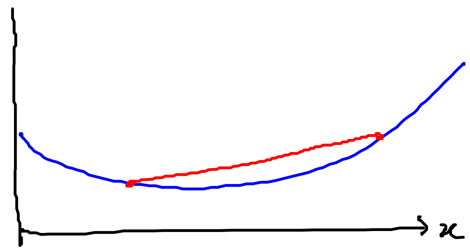






Wednesday, September 22, 2010
10:54 AM

Jensen's inequality

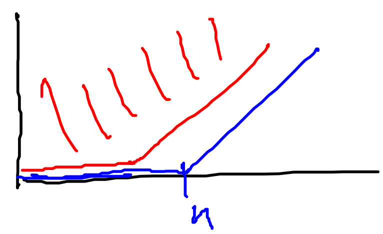


$$\mathbb{E}[f(X)] \geq f(\mathbb{E}[X])$$

$$V_0 = e^{-rT}\mathbb{E}^{Q}[(A_T - K)_+] \qquad f(x) = (x-K)_+$$

$$\geq e^{-rT}(\mathbb{E}^{Q}[A_T] - K)_+$$

$$= e^{-rT}(e^{rT}A_0 - K)_+ = (A_0 - Ke^{-rT})_+$$



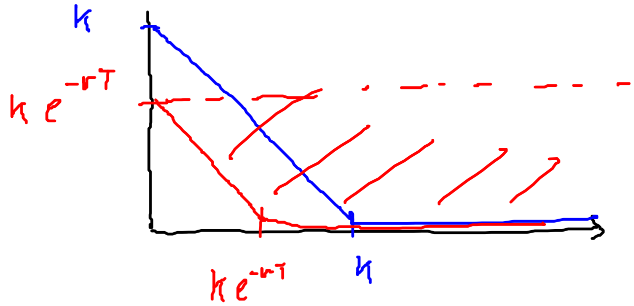
American calls never optimal to exercise early.
(no dividends)

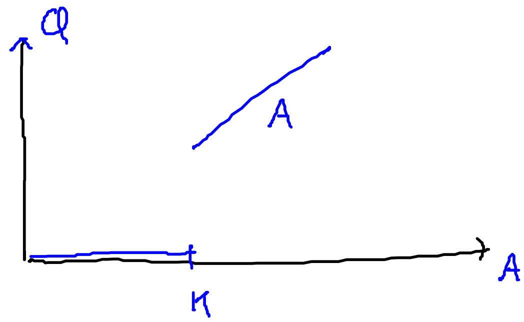
$$V_0 = e^{-rT}\mathbb{E}^{Q}[(K - A_T)_+]$$

$$\geq e^{-rT}(K - \mathbb{E}^{Q}[A_T])_+$$

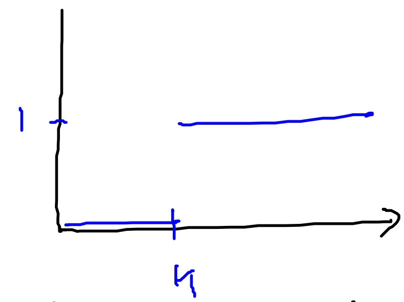
$$= e^{-rT}(K - e^{rT}A_0)_+$$

$$= (Ke^{-rT} - A_0)_+$$

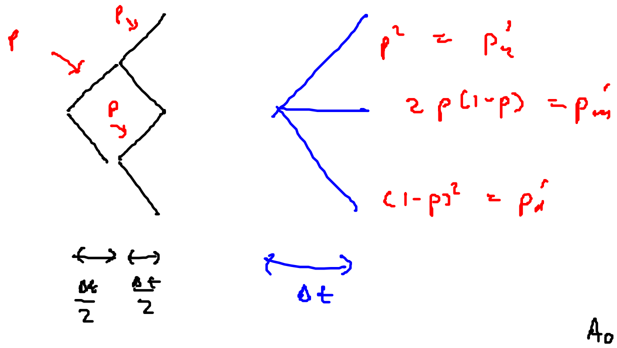




asset-or-nothing



"binary call"
"digital call"



$A_0 \to A_u,\ A_d \qquad 1 \to 1+r,\ 1+r$

$A_d < (1+r) A_0 < A_u$

$A_0 \to A_u,\ A_m,\ A_d \qquad 1 \to 1+r,\ 1+r,\ 1+r$

$A_d < A_m < A_u$ is assumed

$$\boxed{A_d < (1+r) A_0 < A_u \iff \text{no arb}}$$

$A_0 \to A_u,\ A_m,\ A_d \qquad B_0 \to B_u,\ B_m,\ B_d$

if $B_1 > 0$ a.s.

then $\tilde{A}_t = \dfrac{A_t}{B_t}$

$\dfrac{A_0}{B_0} \to A_u/B_u,\ A_m/B_m,\ A_d/B_d \qquad 1 \to 1,\ 1,\ 1$

$$\boxed{\min(\tilde{A}_1) < \frac{A_0}{B_0} < \max(\tilde{A}_1) \iff \text{no arb.}}$$

$\alpha A_0 + \beta$

$$A_0 \begin{cases} A_u \\ A_m \\ A_d \end{cases} \qquad 1 \begin{cases} 1+r \\ 1+r \\ 1+r \end{cases} \qquad \overset{?}{\underset{\shortparallel}{}} \; B_0 \begin{cases} B_u = \alpha A_u + \beta(1+r) \\ B_m = \alpha A_m + \beta(1+r) \\ B_d = \alpha A_d + \beta(1+r) \end{cases}$$

α $\qquad\qquad$ β

$$\underbrace{100 \begin{cases} 110 \\ 100 \\ 90 \end{cases} \qquad 1 \begin{cases} 1 \\ 1 \\ 1 \end{cases}}_{\text{Search for } Q!} \qquad \begin{cases} 10 \\ 50 \\ 20 \end{cases}$$

$$100 = (110\, q_u + 100\, q_m + 90\, q_d)$$

$$q_u + q_m + q_d = 1$$

$$\Rightarrow \quad 0 = 10\, q_u - 10\, q_d \quad \Rightarrow \quad q_u = q_d = q$$

$$\Rightarrow \quad q_m = 1 - 2q$$

$$q_u > 0, \quad q_m > 0, \quad q_d > 0 \quad \Leftrightarrow \quad \text{no arb.}$$

$$q > 0 \qquad q < \tfrac{1}{2}$$

$$q \in (0, \tfrac{1}{2})$$

So $\exists$ a Q $\Rightarrow$ $\exists$ no arb.

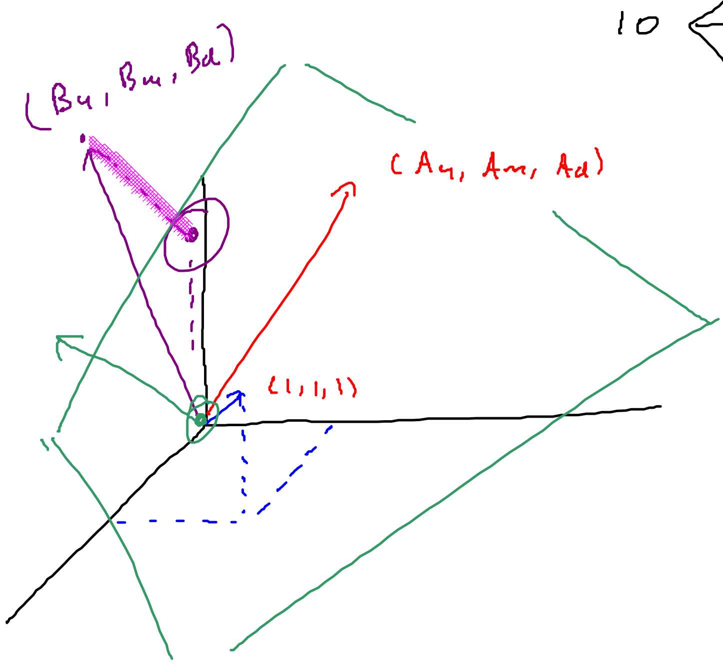
$$\begin{aligned} B_0 &= 10q + 50(1-2q) \\ &\quad + 20q \\ &= 50 - 70q \end{aligned} \qquad 15 < B_0 < 50$$

$$\begin{cases} 50 \\ 10 \\ 20 \end{cases}$$

$$B_0^* = 50 - 70q$$
$$\Rightarrow q = (50 - B_0^*)/70$$

When $\exists$ no unique Q then the market is said to be incomplete.

$$10 \begin{cases} 20 \\ 10 \\ 0 \end{cases} \qquad \begin{aligned} &20q \\ &+ (1-2q)10 \\ &+ q \cdot 0 \end{aligned}$$



$$(\alpha_*, \beta_*) = \underset{(\alpha,\beta)}{\arg\min}\ \mathbb{E}^{P}\left[\left(\alpha A_1 + \beta(1+\overset{r\Delta t}{r}) - B_1\right)^2 \right]$$

$$= \underset{(\alpha,\beta)}{\arg\min} \underbrace{\Big(\alpha^2 \mathbb{E}[A_1^2] + \beta^2(1+r)^2 + \mathbb{E}[B_1^2] + 2\alpha\beta(1+r)\mathbb{E}[A_1] - 2\alpha \mathbb{E}[A_1 B_1] - 2\beta(1+r)\mathbb{E}[B_1] \Big)}_{h(\alpha,\beta)}$$

$$0 = \partial_\alpha h = 2\alpha\, \mathbb{E}[A_1^2] + 2\beta(1+r)\mathbb{E}[A_1] - 2\mathbb{E}[A_1 B_1]$$

$$0 = \partial_\beta h = 2\beta(1+r)^2 + 2\alpha(1+r)\mathbb{E}[A_1] - 2(1+r)\mathbb{E}[B_1]$$

$$\alpha\, \mathbb{E}[A_1^2] + \beta(1+r)\mathbb{E}[A_1] = \mathbb{E}[A_1 B_1]$$

$$\alpha\, \mathbb{E}[A_1] + \beta(1+r) = \mathbb{E}[B_1]$$

$$\alpha\left(\mathbb{E}[A_1^2] - \mathbb{E}[A_1]^2\right) = \mathbb{E}[A_1 B_1] - \mathbb{E}[A_1]\mathbb{E}[B_1]$$

$$\Rightarrow \quad \alpha_* = \frac{\overset{\rho_{AB}\,\sigma_A\,\sigma_B\,\Delta t\, A_0 B_0}{\mathbb{C}[A_1, B_1]}}{}$$

$$\mathbb{V}[A_1] \rightsquigarrow \sigma_A^2 A_0^2 \Delta t$$

$$\beta_* = \frac{1}{1+r}\Big(\underbrace{\mathbb{E}[B_1]}_{B_0(1+\mu_B \Delta t)} - \frac{\mathbb{C}[A_1, B_1]}{\mathbb{V}[A_1]} \underbrace{\mathbb{E}[A_1]}_{A_0(1+\mu_A \Delta t)} \Big)$$

$$\alpha_* = \rho_{AB} \frac{\sigma_B}{\sigma_A} \frac{B_0}{A_0}$$

$$\beta_* = \frac{B_0}{1+r}\Big(1 + \mu_B \Delta t - \rho_{AB} \frac{\sigma_B}{\sigma_A} \cdot (1 + \mu_A \Delta t) \Big)$$

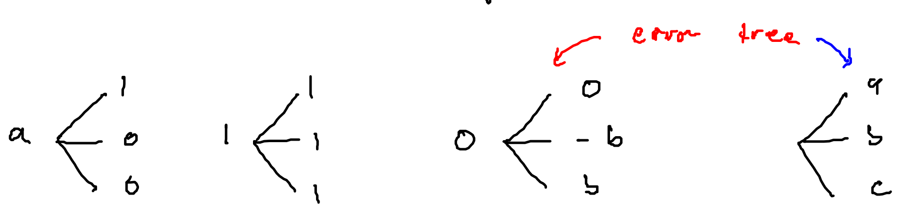
$$= B_0 (1 - r\Delta t)\Big(1 - \rho_{AB}\frac{\sigma_B}{\sigma_A} + \Big(\mu_B - \rho_{AB}\frac{\sigma_B}{\sigma_A}\mu_A\Big)\Delta t \Big)$$

$$\sim B_0 \Big(1 - \rho_{AB}\frac{\sigma_B}{\sigma_A} + \Big((\mu_B - r) - \rho_{AB}\frac{\sigma_B}{\sigma_A}(\mu_A - r)\Big)\Delta t \Big)$$

$$V_0 = \alpha_* A_0 + \beta_*$$

$$= B_0 \Big(1 + \underbrace{\Big((\mu_B - r) - \rho_{AB}\,\sigma_B \frac{(\mu_A - r)}{\sigma_A}\Big)}_{\text{CAPM modified drift}} \Delta t \Big)$$

in the new coordinate system...



to make $\perp$ to A & M $\Rightarrow a = 0$, $b = -c$

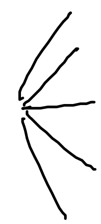
$a = q_u \qquad q_u + q_m + q_d = 1 \qquad 0 = -b\, q_m + b\, q_d$

$\Rightarrow q_m = q_d \in (0,1)$

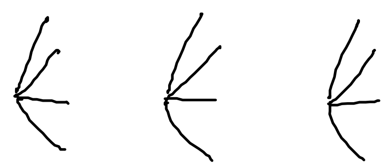
$\therefore$ since $\exists$ a $\mathbb{Q}$ there is no arb.

Variance Minimizing Hedge.

multinomial model



in general Q is not unique even if it exists!!!



if # states > # of traded assets

then Q is not unique in general