Indifference Price

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Binomial Model :

$$A_0 \begin{cases} \xrightarrow{p} A_u \\ \xrightarrow{1-p} A_d \end{cases}$$

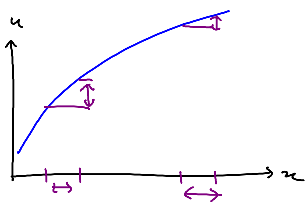
Tree

individual legs - branches

$$A_0 \stackrel{?}{=} \frac{\mathbb{E}[\,A_1\,]}{1+r} = (p\,A_u + (1-p)\,A_d)\,\frac{1}{1+r}$$

utility function :

- $u : \mathbb{R} \mapsto \mathbb{R}$
- increasing
- concave



if L_1 and L_2 are random outcomes

e.g. pay 1, get either 10 or loss 20

$$L_1 = 10\,x - 20\,(1-x) - 1$$

$x \sim$ Bernoulli with prob of p.

$$L_1 \lesssim L_2 \iff \mathbb{E}[u(L_1)] \le \mathbb{E}[u(L_2)]$$

(prefer L_2 to L_1)

Indifference Price or Certainty Equivalent Price:

x – is current wealth

L_1 - do nothing $\rightarrow$ put x in savings account

$$x \begin{cases} x(1+r) \\ x(1+r) \end{cases}$$

L_2 - buy asset & Liquidate

$(x - A_0)$ - in cash

1 unit of asset

$$x \begin{cases} A_u + (x - A_0)(1+r) \\ A_d + (x - A_0)(1+r) \end{cases}$$

$$\mathbb{E}[\,u(L_1)\,] = \mathbb{E}[\,u(L_2)\,]$$

exponential utility $\quad u(x) = \dfrac{1-e^{-\gamma x}}{\gamma} \to -\dfrac{1}{\gamma} e^{-\gamma x}$

$$\xrightarrow[\gamma \downarrow 0]{} x$$

$$-\frac{1}{\gamma} e^{-\gamma x(1+r)} = -\frac{1}{\gamma}\Big\{ p\, e^{-\gamma[A_u + (x - A_0)(1+r)]} + (1-p)\, e^{-\gamma[A_d + (x-A_0)(1+r)]}\Big\}$$

$$\Rightarrow \quad 1 = e^{\gamma A_0 (1+r)} \left(p\, e^{-\gamma A_u} + (1-p)\, e^{-\gamma A_d} \right)$$

$$\boxed{A_0 = -\frac{1}{\gamma(1+r)} \ln\left(p\, e^{-\gamma A_u} + (1-p)\, e^{-\gamma A_d} \right)}$$

$$= -\frac{1}{\gamma(1+r)} \left[\ln\left(p\, e^{-\gamma(A_u - A_d)} + (1-p)\right) - \gamma A_d \right]$$

$$= -\frac{1}{\gamma(1+r)} \ln\left((1-p) + p\, e^{-\gamma(A_u - A_d)}\right) + \frac{A_d}{1+r}$$

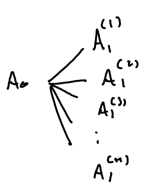
$$\xrightarrow[\gamma \uparrow +\infty]{} \frac{A_d}{1+r}$$

$\ln(1+y) = y + o(y)$

$$\approx -\frac{1}{\gamma(1+r)} \ln\Big(\underbrace{p(1-\gamma A_u) + (1-p)(1-\gamma A_d)}_{1 - \gamma[p A_u + (1-p) A_d]} + o(\gamma) \Big)$$

$$\approx -\frac{1}{\gamma(1+r)} \left(-\gamma (p A_u + (1-p) A_d) + o(\gamma) \right)$$

$$\xrightarrow[\gamma \downarrow 0]{} \frac{1}{1+r}\left(p A_u + (1-p) A_d \right) = \frac{1}{1+r} \mathbb{E}[A_1]$$



branching probs are $p^{(i)}$, $\sum_i p^{(i)} = 1$

$0 < p^{(i)} < 1$

$$A_0 = -\frac{1}{\gamma(1+r)} \ln \sum_i p^{(i)} e^{-\gamma A_1^{(i)}}$$

$$= -\frac{1}{\gamma(1+r)} \ln \mathbb{E}\left[e^{-\gamma A_1} \right]$$

Replication

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$$(1, A_0, B_0) \begin{array}{l} \nearrow^{p} \; ((1+r), A_u, B_u) \\ \searrow_{1-p} \; ((1+r), A_d, B_d) \end{array}$$

$$A_0 \begin{array}{l} \nearrow A_u \\ \searrow A_d \end{array} \qquad B_0 \begin{array}{l} \nearrow B_u \\ \searrow B_d \end{array} \qquad 1 \begin{array}{l} \nearrow 1+r \\ \searrow 1+r \end{array}$$

$$(1, 100, 10) \begin{array}{l} \nearrow (1, 110, 5) \\ \searrow (1, 90, 12) \end{array}$$

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replicate asset's B behavior by trading in asset A & MM

$$\underset{\alpha}{A_0 \begin{array}{l} \nearrow A_u \\ \searrow A_d \end{array}} + \underset{\beta}{1 \begin{array}{l} \nearrow 1+r \\ \searrow 1+r \end{array}} \rightarrow A_0 \alpha + \beta \begin{array}{l} \nearrow \alpha A_u + (1+r)\beta = B_u \\ \searrow \alpha A_d + (1+r)\beta = B_d \end{array}$$

$A_0 \alpha + \beta = B_0$ to avoid arbitrage!

$$\Rightarrow \quad B_0 = \frac{1}{1+r}\left(q \, B_u + (1-q) \, B_d \right)$$

where
$$q = \frac{(1+r) A_0 - A_d}{A_u - A_d}$$

$0 < q < 1$ iff $\exists$ no arb.

(no arb. $\Leftrightarrow$ $A_d < (1+r) A_0 < A_u$)

$A_d \quad (1+r)A_0 \quad A_u$

$$\boxed{B_0 = \frac{1}{1+r}\,\mathbb{E}^{\mathbb{Q}}[\,B_1\,]}$$

$\mathbb{Q}(\,B_1 = B_u) = q,$

$\mathbb{Q}(\,B_1 = B_d) = 1-q$

$\hookrightarrow$ risk-neutral probability

b/c $\mathbb{E}^{\mathbb{Q}}[\,B_1] = (1+r)\,B_0$, i.e. under the measure $\mathbb{Q}$, B grows at the risk-free rate!

$$A_0(1+r) = \mathbb{E}^{\mathbb{Q}}[\,A_1] = q\,A_u + (1-q)\,A_d$$

$$= q\,(A_u - A_d) + A_d$$

$$\Rightarrow \quad q = \frac{A_0(1+r) - A_d}{A_u - A_d}$$

No arbitrage

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Intuition of an arbitrage is a strategy s.t. you make a "riskless profit" beyond the risk-free rate.

Let V_t be the value of a strategy at time t

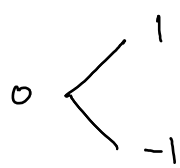
V is an arbitrage if the following hold:

i) $V_0 = 0$ (cost nothing)

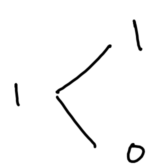
ii) $\exists$ a t s.t.

a. $\mathbb{P}(V_t \geq 0) = 1$ (never lose)

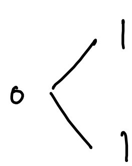
b. $\mathbb{P}(V_t > 0) > 0$ (sometimes win)



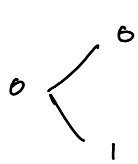
not an arb



not an arb



is an arb



is an arb

$$\begin{tikzcd}[row sep=small] & 0 \\ 0 \arrow[ur, dash] \arrow[dr, dash] & \\ & 0 \end{tikzcd}$$

is not arb

$$\begin{tikzcd}[row sep=small] & 0 & & & 0 \\ 0 \arrow[ur, dash] \arrow[dr, dash] & & \xrightarrow[\text{short}]{} & 0 \arrow[ur, dash] \arrow[dr, dash] & \\ & -1 & & & 1 \end{tikzcd}$$

No Arbitrage continued

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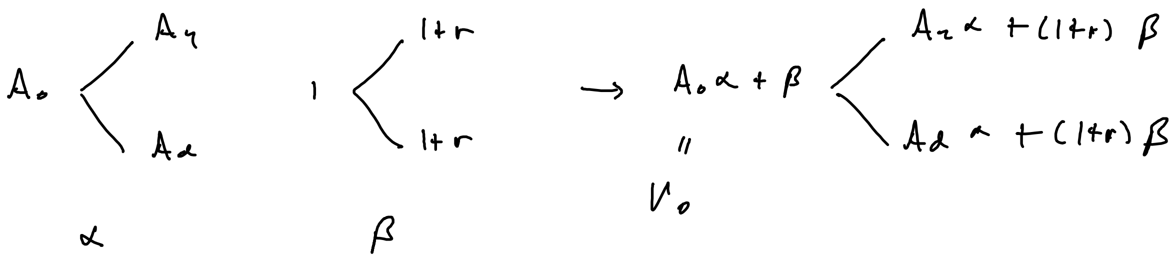

$$V_0 = 0 \Rightarrow \beta = -A_0 \alpha$$

$$V_1^{(u)} = \alpha \left(A_u - (1+r) A_0 \right)$$

$$V_1^{(d)} = \alpha \left(A_d - (1+r) A_0 \right)$$

to avoid arb must have either:

I. $A_u - (1+r) A_0 > 0$

& $A_d - (1+r) A_0 < 0$

$\Leftrightarrow$ $\boxed{A_d < (1+r) A_0 < A_u}$

II. $\cancel{A_u < (1+r) A_0 < A_d}$

we need $A_u > A_d$

Can you prove that $A_0 = -\frac{1}{\gamma(1+r)} \ln \mathbb{E}\left[e^{-\gamma A_1} \right]$

provides an arbitrage free price?

Numeraires

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$$\frac{B_0}{1} = \mathbb{E}^{\mathbb{Q}}\left[\frac{B_1}{1+r}\right]$$

$$\tilde{B}_0 = \mathbb{E}^{\mathbb{Q}}[\tilde{B}_1] \qquad \tilde{B}_t = \frac{B_t}{M_t} \quad \text{relative price}$$

$\tilde{B}$ is a "fair-game" under measure $\mathbb{Q}$.

$$A_0 = \frac{1}{1+r}\left(q\, A_u + (1-q)\, A_d\right)$$

$$\frac{A_0}{C_0} = \frac{1}{M_u}\frac{q}{C_0} A_u + \frac{(1-q)}{M_d\, C_0} A_d$$

want ...

$$(\quad)\frac{A_u}{C_u} + (\quad)\frac{A_d}{C_d}, \quad \tilde{A}_0 = \mathbb{E}^{\mathbb{Q}^c}[\tilde{A}_1]$$

$$= \underbrace{\left(\frac{C_u/C_0}{M_u/M_0}\right) q}_{q^*} \frac{A_u}{C_u} + \underbrace{\left(\frac{C_d/C_0}{M_d/M_0}\right)(1-q)}_{q^{**}} \frac{A_d}{C_d}$$

if $C_u \;\&\; C_0 > 0, \quad q^* > 0$

$C_d \;\&\; C_0 > 0, \quad q^{**} > 0$

$$q^* + q^{**} = \frac{M_0}{C_0}\left[\frac{C_u}{M_u} q + \frac{C_d}{M_d}(1-q)\right]$$

$$= \frac{M_0}{C_0} \frac{C_0}{M_0} = 1$$

$$\therefore \boxed{\frac{A_0}{C_0} = \mathbb{E}^{\mathbb{Q}^C}\left[\frac{A_1}{C_1}\right]}$$

$$\mathbb{Q}^C(A_1 = A_u) = \left(\frac{C_u/C_0}{M_u/M_0}\right) \cdot q$$

$$= \left(\frac{C_u/C_0}{M_u/M_0}\right) \mathbb{Q}(A_1 = A_u)$$

$$\mathbb{Q}^C(A_1 = A_d) = \left(\frac{C_d/C_0}{M_d/M_0}\right)(1-q)$$

$$= \left(\frac{C_d/C_0}{M_d/M_0}\right) \mathbb{Q}(A_1 = A_d)$$

Numeraire assets are traded assets that are > 0 a.s. (C)

no arb iff $\exists$ a measure $\mathbb{Q}^C$ s.t.

$$\mathbb{E}^{\mathbb{Q}^C}[\tilde{A}_1] = \tilde{A}_0 \qquad \tilde{A}_t = \frac{A_t}{C_t}$$

For all traded assets.