IMPA Commodities Course : Spot Models

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Table of contents

What will you learn today?

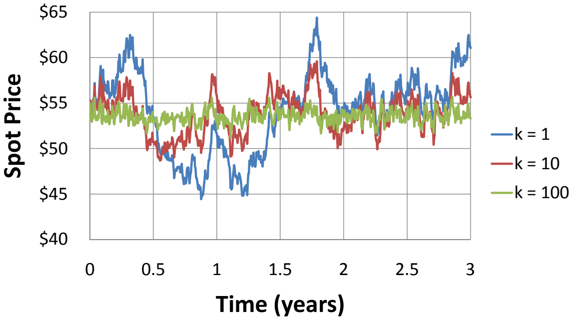
- Spot Models
- Implied Forward Prices
- Options on Spots
- Options on Forwards

Schwartz's One-Factor Spot Model

- **Scwartz(1997)** introduced a **mean-reverting** spot model:

$$dS_t = \kappa(\theta - \ln S_t)\, S_t\, dt + \sigma\, S_t\, dW_t$$

W_t is a Wiener process under the **real-world measure** $\mathbb{P}$.



Schwartz's One-Factor Spot Model

- In this model, prices are **log-normal**

$$\begin{aligned} S_{t+\Delta t} &= \exp\Big\{\theta' + (\ln(S_t) - \theta')e^{-\kappa\,\Delta t} \\ &\qquad + \sigma \int_t^{t+\Delta t} e^{-\kappa(t+\Delta t-u)}\,dW_u\Big\} \end{aligned}$$

where $\theta' = \theta - \frac{1}{2}\sigma^2$.

Schwartz's One-Factor Spot Model

- The **mean** of log-spot prices is

$$\mathbb{E}_t^{\mathbb{P}}[\ln(S_{t+\Delta t}/S_t)] \quad = \quad \theta' + (\ln(S_t) - \theta')e^{-\kappa\,\Delta t}$$

- The **variance** of log-spot prices is

$$Var_t^{\mathbb{P}}[\ln(S_{t+\Delta t}/S_t)] \quad = \quad \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa\Delta t}\right)$$

- Notice that the mean and variance are bounded for all time
- **Invariant distribution** of log-prices is normal with mean$= \theta'$ & variance$= \sigma^2/2\kappa$

One-Factor Spot Model: Induced Forward Prices

- **Forward prices** with **stock underliers** are given by $F_t(T) = \mathbb{E}_t^{\mathbb{Q}}[S_T]$ where drift of S_t under $\mathbb{Q}$ is r
- **Forward prices** with **commodity underliers** must incorporate:
  - P.V. of **storage costs** C in the Forward price for the **buyer**:

$$F_t(T) \leq (S_t + C)\, e^{r(T-t)}$$

  - P.V. of premium (or **convenience yield**) for giving up the commodity in the Forward price for the **seller**:

$$F_t(T) \geq (S_t + C)\, e^{r(T-t)} - Y$$

One-Factor Spot Model: Induced Forward Prices

- Can satisfy bounds by writing

$$F_t(T) = S_t \exp\left\{ \int_t^T (r + c_t(s) - y_t(s))\, ds \right\}$$

- In general, introduce a new measure $\mathbb{Q}$ induced by the **Radon-Nikodym derivative** process:

$$\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_t \triangleq \exp\left\{ -\frac{1}{2}\int_0^t \lambda_s\, ds + \int_0^t \lambda_s\, dW_s \right\}$$

- Then $\overline{W}_t \triangleq \int_t^T \lambda_s ds + W_t$ is a $\mathbb{Q}$ Wiener process and

$$dS_t = [\mu_t - \sigma\,\lambda_t]\, S_t\, dt + \sigma\, S_t\, d\overline{W}_t$$

- λ_s is the **market price of risk**

One-Factor Spot Model: Induced Forward Prices

- For Schwartz model, choosing $\lambda_s = \lambda_0 + \lambda_1 \ln S_t$, maintains mean-reverting model class with new parameters

$$dS_t = \overline{\kappa}(\overline{\theta} - \ln S_t)\, S_t\, dt + \sigma\, S_t\, d\overline{W}_t$$

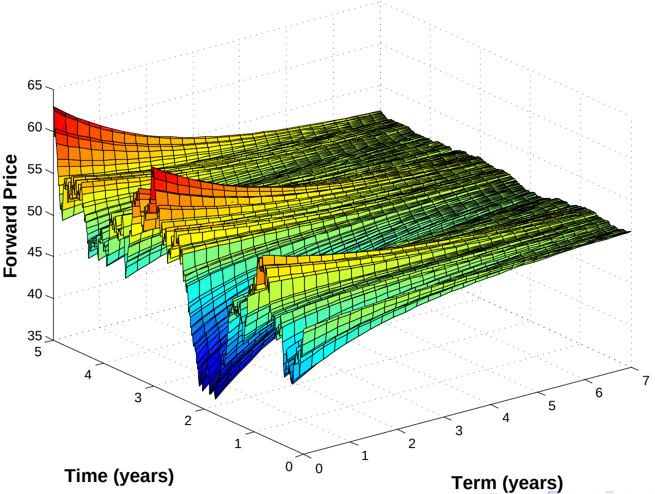
- **Forward prices** are then easily obtained

$$F_t(T) \triangleq \mathbb{E}_t^{\mathbb{Q}}[S_T]$$

$$= \exp\left\{ \overline{\theta}' + (\ln(S_t) - \overline{\theta}')e^{-\overline{\kappa}(T-t)} + \frac{\sigma^2}{4\overline{\kappa}}\left(1 - e^{-2\overline{\kappa}(T-t)}\right) \right\}$$

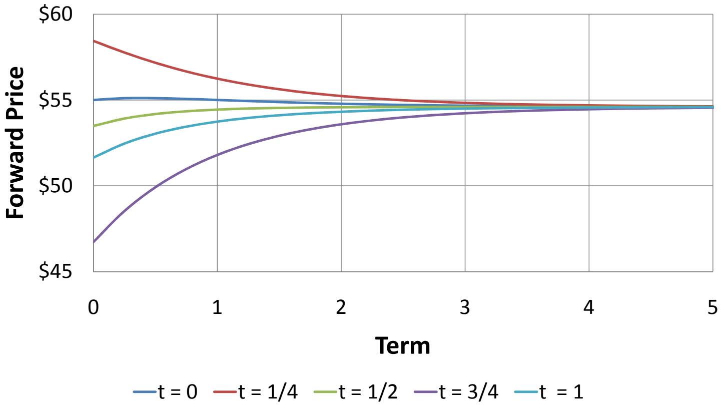
One-Factor Spot Model: Induced Forward Prices

Sample path of forward price curves:



One-Factor Spot Model: Induced Forward Prices

Forward curves at quarterly time intervals:



One-Factor Spot Model: Induced Forward Prices

- For a **fixed maturity date**, forward prices are exponential martingales:

$$\frac{dF_t(T)}{F_t(T)} = \sigma\, e^{-\overline{\kappa}(T-t)}\, d\overline{W}_t$$

- For a **fixed term**, forward prices (i.e. $X_t \triangleq F_t(t+\tau)$) are mean-reverting:

$$dX_t = \overline{\kappa}(h_t - X_t)\, dt + \sigma\, e^{-\overline{\kappa}\,\tau}\, d\overline{W}_t$$

  where h_t is a deterministic function of time.

- Notice as T (or τ) $\to \infty$ the vol tends to zero

One-Factor Spot Model: Options on Spot

- **Call Option** on Spot:

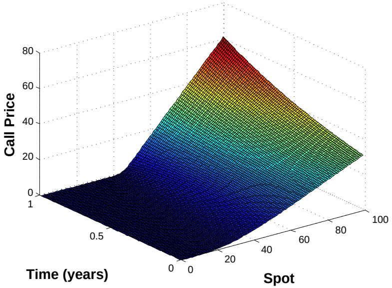
$$\begin{aligned} C_t &= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-r\tau}\,(S_T - K)_+\right] \\ &= S_t\, e^{(\overline{\mu}-r)\tau}\,\Phi(d_+^*) - K\, e^{-r\tau}\,\Phi(d_-^*) \end{aligned}$$

where

$$\begin{aligned} d_\pm &= \frac{\ln(S/K) + (\overline{\mu} \pm \frac{1}{2}\overline{\sigma}^2)\tau}{\sqrt{\overline{\sigma}^2\,\tau}} \\ \overline{\sigma}^2 &= \frac{\sigma^2}{2\overline{\kappa}\,\tau}(1 - e^{-2\overline{\kappa}\tau}) \\ \overline{\mu} &= \frac{1}{\tau}\left\{(\overline{\theta}' - \ln(S_t))(1 - e^{-\overline{\kappa}\tau}) + \tfrac{1}{2}\overline{\sigma}^2\right\} \end{aligned}$$

One-Factor Spot Model: Options on Spot

- **Call Option** on Spot:



(a) $\kappa = 0.50$

One-Factor Spot Model: Options on Forwards

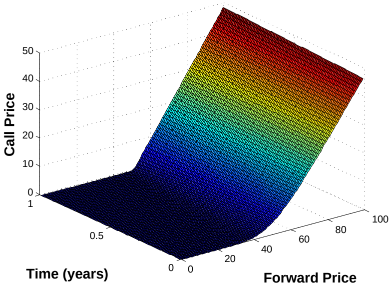
- **Call Option** on Forward:

$$
\begin{aligned}
C_t &= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-r\tau}\left(F_T(U)-K\right)_+\right] \\
&= e^{-r\,\tau}\left\{F_t(U)\,\Phi(d_+^*) - K\,\Phi(d_-^*)\right\}
\end{aligned}
$$

$$
\begin{aligned}
\text{where} \quad d_\pm^* &= \frac{\ln(F_t(U)/K) \pm \frac{1}{2}(\sigma^*)^2\tau}{\sqrt{(\sigma^*)^2\,\tau}} \\
(\sigma^*)^2 &= \frac{\sigma^2}{2\overline{\kappa}\,\tau}\left(e^{-2\overline{\kappa}(U-T)} - e^{-2\overline{\kappa}(U-t)}\right)
\end{aligned}
$$

One-Factor Spot Model: Options on Forwards

- **Call Option** on Forward:



(b) $\kappa = 0.50$

One-Factor Spot Model: Options on Forwards

- **Calender Spread Option** on Forward:

$$
\begin{aligned}
C_t &= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-r\tau}\left(F_T(U_1) - F_T(U_2)\right)_+\right] \\
&= e^{-r\tau}\left\{F_t(U_1)\,\Phi(d_+^\dagger) - F_t(U_2)\,\Phi(d_-^\dagger)\right\}
\end{aligned}
$$

$$
\begin{aligned}
\text{where} \quad d_\pm^\dagger &= \frac{\ln(F_t(U_1)/F_t(U_2)) \pm \frac{1}{2}(\sigma^\dagger)^2\tau}{\sqrt{(\sigma^\dagger)^2\,\tau}} \\
(\sigma^\dagger)^2 &= \frac{\sigma^2}{2\overline{\kappa}\,\tau}(e^{-\overline{\kappa}(U_2-T)} - e^{-\overline{\kappa}(U_1-T)})^2(1 - e^{-2\overline{\kappa}(T-t)})
\end{aligned}
$$

One-Factor Spot Model: Options on Forwards

- **Calender Spread Option** on Forward **with cost**:

$$C_t = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-r\tau}\left(F_T(U_1) - F_T(U_2) - K\right)_+\right]$$

- Try it!

Two-Factor Spot Models: Pilipovic

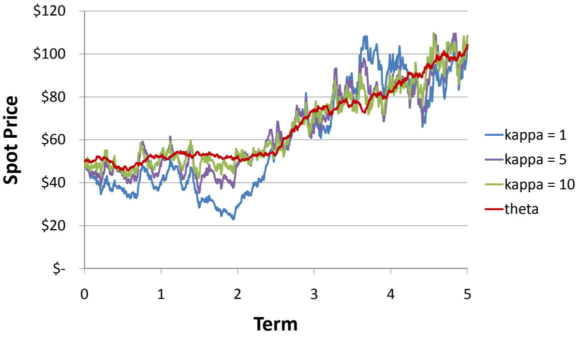
- One-factor models are only useful in the short term and do not match forward curves well
- **Pilipovic**(1997) introduced the following model to correct for this

$$\begin{aligned} dS_t &= \kappa(\theta_t - S_t)\, dt + \sigma\, S_t\, dW_t^{(1)} \\ d\theta_t &= \theta_t \left(\mu\, dt + \eta\, dW_t^{(2)}\right) \end{aligned}$$

  with $W_t^{(1)}$ and $W_t^{(2)}$ uncorrelated Wiener processes
- **Xu**(2004) generalized this model by incorporating seasonality, making σ time dependent and θ_t an OU process

Two-Factor Spot Models: Pilipovic

- Sample paths in the Pilipovic model:



Two-Factor Spot Models: Pilipovic

- Can solve the system of SDEs to find

$$S_t = h_t \left(S_0 \, e^{-\kappa t} + \kappa \int_0^t \theta_u \, e^{-\kappa(t-u)} h_u^{-1} \, du \right)$$

$$h_t = \exp\left\{ -\tfrac{1}{2}\sigma^2 t + \sigma W_t^{(1)} \right\}$$

$$\theta_t = \theta_0 \exp\left\{ (\mu - \tfrac{1}{2}\eta^2) t + \eta W_t^{(2)} \right\}$$

- Can obtain Forward Prices, but **distribution** of S_t is **not known**

Two-Factor Spot Models: HJ

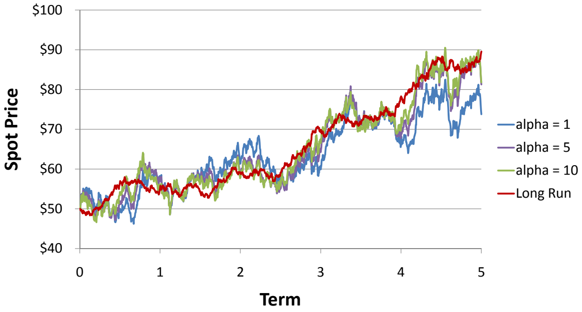
- **Barlow, Gusev, and Lai**(2004) and **Hikspoors & Jaimungal**(2007) introduced a more tractable generalization as follows

$$
\begin{aligned}
S_t &= \exp\{g_t + X_t\} \\
dX_t &= \alpha\,(Y_t - X_t)\,dt + \sigma\,dW_t^{(1)} \\
dY_t &= \beta\,(\phi - Y_t)\,dt + \eta\,dW_t^{(2)}
\end{aligned}
$$

$W_t^{(1)}$ and $W_t^{(2)}$ are correlated Wiener processes, g_t incorporates seasonality, and σ can easily be made deterministic

Two-Factor Spot Models: HJ

- Sample paths in the HJ model:



Two-Factor Spot Models: HJ

- Can show that

$$
\begin{aligned}
Y_t &= \phi + (Y_0 - \phi)\, e^{-\beta\, t} + \eta \int_0^t e^{-\beta\,(t-u)}\, dW_u^{(2)}\,; \\
X_t &= G_{0,t} + e^{-\alpha t} X_0 + M_{0,t}\, Y_0 \\
&\qquad + \sigma \int_0^t e^{-\alpha t}\, dW_u^{(1)} + \eta \int_0^t M_{u,t}\, dW_u^{(2)}\,,
\end{aligned}
$$

where

$$
\begin{aligned}
M_{s,t} &= \frac{\alpha}{\alpha - \beta} \left(e^{-\beta\,(t-s)} - e^{-\alpha\,(t-s)} \right) \\
G_{s,t} &= \phi(1 - e^{-\alpha(t-s)}) - \phi M_{s,t}
\end{aligned}
$$

- Distribution of S_t is log-normal

Two-Factor Spot Model: Induced Forward Prices

- Forward prices $F_t(T)$ in the HJ model satisfy the PDE

$$\begin{cases} (\partial_t + \mathcal{L})\, F &= 0\,, \\ F_T(T) &= e^x\,. \end{cases}$$

- $\mathcal{L}$ is the infinitesimal generator of the processes (X_t, Y_t):

$$\mathcal{L} = \alpha(y-x)\partial_x + \beta(\phi - y)\partial_y + \tfrac{1}{2}\sigma^2\partial_{xx} + \tfrac{1}{2}\eta^2\partial_{yy} + \rho\eta\sigma\partial_{xy}$$

Two-Factor Spot Model: Induced Forward Prices

- This is an **affine model** so that

$$F_t(T) = \exp\{a_t(T) + b_t(T)\,X_t + c_t(T)\,Y_t\}$$

for deterministic functions $a_t(T)$, $b_t(T)$ and $c_t(T)$ of t. Subject to

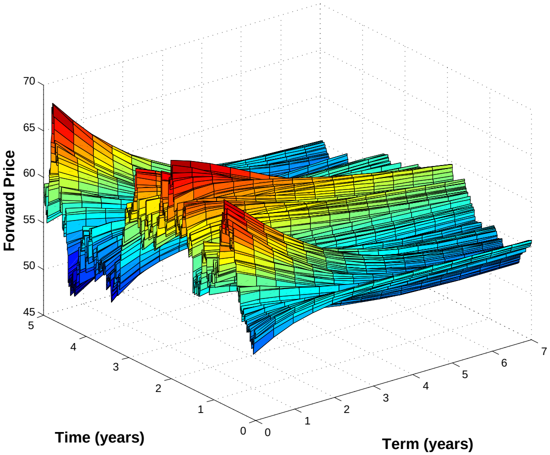
$$a_T(T) = 0\ , \quad b_T(T) = 1\ , \quad c_T(T) = 0$$

- These functions satisfy the coupled ODEs

$$\begin{cases} \partial_t b - \alpha\, b &= 0\ , \\ \partial_t c - \beta\, c + \alpha\, b &= 0\ , \\ \partial_t a + \phi\beta\, c + \frac{1}{2}\sigma^2\, b^2 + \frac{1}{2}\eta^2\, c^2 + \sigma\eta\rho\, b\, c &= 0\ . \end{cases}$$

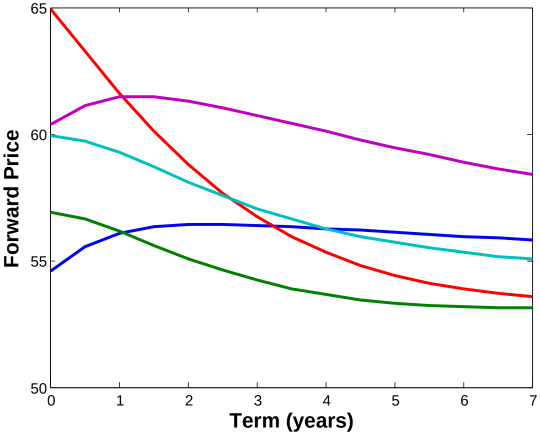
Two-Factor Spot Model: Induced Forward Prices

Sample path of forward price curves in the 2-factor HJ model



Two-Factor Spot Model: Induced Forward Prices

forward price curves in the 2-factor HJ model



Two-Factor Spot Model: Option Prices

- Forward prices for fixed maturity $F_t(T)$ are once again exponential martingales

$$\frac{dF_t(T)}{F_t(T)} = \sigma\, b_t\, dW_t^{(1)} + \eta\, c_t\, dW_t^{(2)}$$

- Of course, both risk factors feed into the dynamics

Two-Factor Spot Model: Option Prices

- Call option on Forward

$$C_t = e^{-r\,\tau}\left\{F_t(U)\,\Phi(d_+^*) - K\,\Phi(d_-^*)\right\}$$

where

$$d_\pm^* = \frac{\ln(F_t(U)/K) \pm \frac{1}{2}(\sigma^*)^2\tau}{\sqrt{(\sigma^*)^2\,\tau}}$$

$$(\sigma^*)^2 = \frac{1}{\tau}\left\{\left(\sigma^2+\overline{\eta}^2-2\rho\sigma\overline{\eta}\right)g(t,T,U,2\alpha) + \overline{\eta}^2 g(t,T,U,2\beta) - 2\overline{\eta}\left(\overline{\eta}+\rho\sigma\right)g(t,T,U,\alpha+\beta)\right\}$$

and

$$g(t,T,U,a) \triangleq \frac{1}{a}\left(e^{-a(U-T)} - e^{-a(U-t)}\right), \qquad \overline{\eta} = \frac{\alpha}{\alpha-\beta}\eta$$

Stochastic Convenience Yield Models

- **Gibson & Schwartz** (1990) introduced a stochastic convenience yield mode to correct one-factor models
- Spot price S_t is (conditionally) GBM with an OU process driving convenience yield δ_t

$$
\begin{aligned}
dS_t &= S_t \left((r - \delta_t)\, dt + \sigma_1\, dW_t^{(1)} \right) \\
d\delta_t &= [\kappa(\alpha - \delta_t) - \lambda]\, dt + \sigma_2\, dW_t^{(2)}
\end{aligned}
$$

where $W_t^{(1)}$ and $W_t^{(2)}$ correlated Wiener processes.

Stochastic Convenience Yield Models

- **Jamshidian & Fein** (1990) demonstrated that forward prices are **affine**:

$$F_t(T) = S_t \exp\{a_t(T) - b_t(T)\,\delta_t\}$$

where

$$\begin{aligned} a_t(T) \;&=\; \left(r - \alpha + \frac{\lambda}{\kappa} + \frac{1}{2}\frac{\sigma_2^2}{\kappa^2} - \frac{\sigma_1\sigma_2\rho}{\kappa}\right)(T-t) \\ &\quad + \frac{1}{4}\sigma_2^2\,\frac{1-e^{-2\kappa(T-t)}}{\kappa^3} \\ &\quad + \left((\alpha - \tfrac{\lambda}{\kappa})\kappa + \sigma_1\sigma_2\rho - \frac{\sigma_2^2}{\kappa}\right)\frac{1-e^{-\kappa(T-t)}}{\kappa^2} \\ b_t(T) \;&=\; \frac{1-e^{-\kappa(T-t)}}{\kappa} \end{aligned}$$

Stochastic Interest Rates

- **Schwartz (1997)** also extended the model to incorporate stochastic interest rates

$$
\begin{aligned}
dS_t &= S_t\left((r-\delta_t)\,dt + \sigma_S\,dW_t^{(1)}\right) \\
d\delta_t &= [\kappa(\alpha-\delta_t)-\lambda]\,dt + \sigma_\delta\,dW_t^{(2)} \\
dr_t &= \beta(\theta - r_t)\,dt + \sigma_r\,dW_t^{(3)}
\end{aligned}
$$

- This model is also affine and it is possible to solve for forward prices and European option prices explicitly
- For commodities, there is no significant advantage gained by incorporating stochastic interest rates