IMPA Commodities Course : Forward Price Models

Sebastian Jaimungal
sebastian.jaimungal@utoronto.ca

Department of Statistics and
Mathematical Finance Program,
University of Toronto, Toronto, Canada
http://www.utstat.utoronto.ca/sjaimung

February 19, 2008

Table of contents

Basic Forward Price Models

- Basic **Black**(1976) Forward price model assumes

$$\frac{dF_t(T)}{F_t(T)} = \sigma\, dW_t$$

  where W_t is a $\mathbb{Q}$-Wiener process

- Each forward price evolves like a **GBM** on its own
- Option prices are trivial. Here is a **call price**

$$\begin{aligned} C_t &= e^{-r(T-t)} \{F_T(U)\,\Phi(d_+) - K\,\Phi(d_-)\} \\ d_\pm &= \frac{\ln(F_T(U)/K) \pm \frac{1}{2}\sigma^2(T-t)}{\sqrt{\sigma^2(T-t)}} \end{aligned}$$

Multifactor Forward Price Models

- **Multifactor** forward price model assume instead

$$\frac{dF_t(T)}{F_t(T)} = \sum_{k=1}^{K} \sigma_t^{(k)}(T)\, dW_t^{(k)}$$

  where $W_t^{(k)}$ are correlated $\mathbb{Q}$-Wiener process with $d[W_t^{(k)}, W_t^{(l)}] = \rho_{kl}\, dt$.
- The **covariance structure** is estimated from **principle component analysis** (PCA) of forward price curves
- Volatility functions often assumed to be deterministic
- A further simplifying assumption is often made: $\sigma_t^{(k)}(T) = \sigma_t\, \sigma^{(k)}(T - t)$

Multifactor Forward Price Models

- A **principal component analysis** on forward curves is used to determine main factors
- The forward prices at a constant set of terms $\{\tau_1, \tau_2, \dots, \tau_n\}$ are interpolated from given data

$$\begin{pmatrix} F_{t_1}(\tau_1) & F_{t_1}(\tau_2) & F_{t_1}(\tau_3) & \dots & F_{t_1}(\tau_n) \\ F_{t_2}(\tau_1) & F_{t_2}(\tau_2) & F_{t_2}(\tau_3) & \dots & F_{t_2}(\tau_n) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ F_{t_N}(\tau_1) & F_{t_N}(\tau_2) & F_{t_N}(\tau_3) & \dots & F_{t_N}(\tau_n) \end{pmatrix}$$

Multifactor Forward Price Models

- The daily returns are then computed

$$R_{t_i}(\tau_k) = \frac{F_{t_{i+1}}(\tau_k) - F_{t_i}(\tau_k)}{F_{t_i}(\tau_k)}$$

- The resulting time series used to estimate the covariance matrix Σ

$$\begin{aligned}
\widehat{\mu}_k &= \frac{1}{N}\sum_{i=1}^{N} R_{t_i}(\tau_k) \\
\widehat{\Sigma}_{jk} &= \frac{1}{N}\sum_{i=1}^{N} \left(R_{t_i}(\tau_j) - \widehat{\mu}_j\right)\left(R_{t_i}(\tau_k) - \widehat{\mu}_k\right)
\end{aligned}$$

Multifactor Forward Price Models

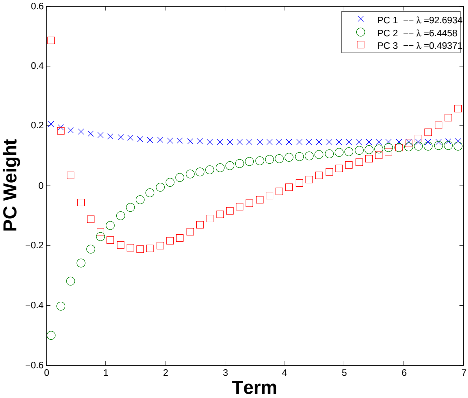
- Given the estimated covariance matrix, obtain eigenvectors $(v_1, v_2, \ldots, v_n)$ and eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_n)$:

$$\Sigma = V \Lambda V^T$$

- The **eigenvectors** corresponding to the **first n-largest eigenvalues** are called the **principal components**
- For crude oil and heating oil, first three components account for 99% of variability:
  - PC-1 accounts for **parallel shifts**
  - PC-2 accounts for **tilting**
  - PC-3 accounts for **bending**
- For electricity more than 10 PCs are required to account for 99% of variability

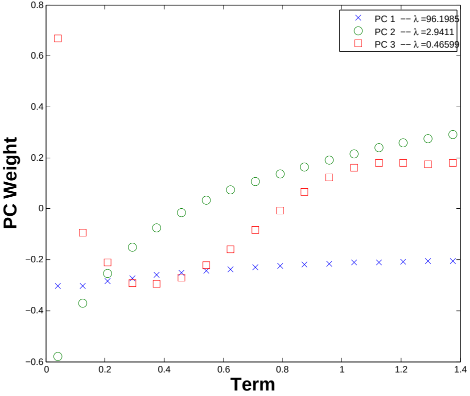
Multifactor Forward Price Models

Crude Oil first 3 Principal Components



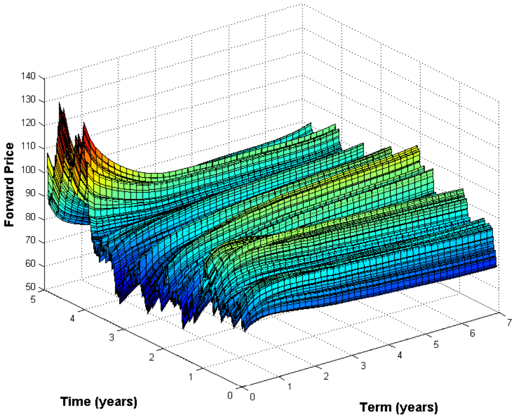
Multifactor Forward Price Models

Heating Oil first 3 Principal Components



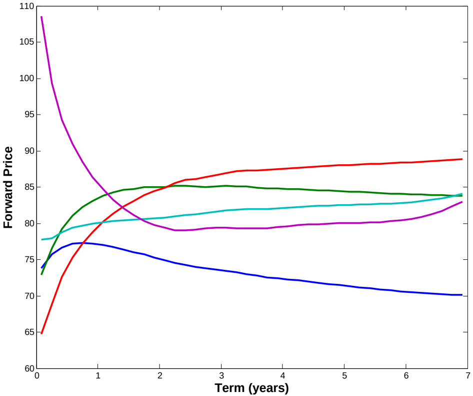
Multifactor Forward Price Models

Simulation of crude oil forward curves using 3 principal components



Multifactor Forward Price Models

Simulation of crude oil forward curves using 3 principal components



Functional PCA

- **Functional Principal Component Analysis**(FPCA) developed by **Ramsay & Silverman** (book in 2005) views the data as sequence of **random functions**
- Allows domain specific knowledge to augment PCs
- Produces smooth PCs
- Allows interpolation and extrapolation between observation points
- Easily handles non-equal spaced and non-equal number of data per curve

Functional PCA

- **Jaimungal & Ng** (2007) introduced a **consistent FPCA** approach appropriate for commodity time-series:
  - Project data onto a functional basis
  - Fit a Vector Autoregressive (VAR) process to time-series coefficients
  - Detrend the coefficients
  - Remove predictable part of VAR process to extract true stochastic degrees of freedom
  - Project covariance matrix onto a distortion metric associated with basis functions
  - Solve a modified eigen-problem for PCs of basis coefficients
  - Weight basis functions according to eigen-vectors

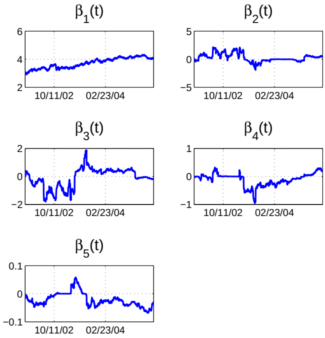
Consistent Functional PCA

- Project data points onto basis functions $\{\phi_1(\tau), \dots, \phi_K(\tau)\}$ for each trading date $t_1, \dots, t_M$

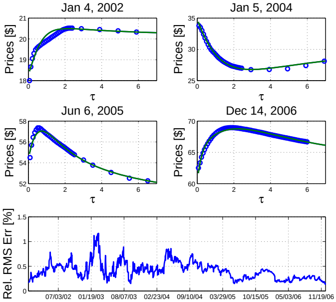
$$
\begin{aligned}
F_{t_m}(t_m + \tau_i) &= \sum_{k=1}^{K} \beta_{m,k}\, \phi_k(\tau_i) \\
\widehat{\beta}_{m,i} &= \arg\min_{\beta_{m,i}} \sum_{i=1}^{N_m} \left\| F^*_{t_m}(t_m + \tau_i) - \sum_{k=1}^{K} \beta_{m,k}\, \phi_k(\tau_i) \right\|^2
\end{aligned}
$$

to produce a time-series of fitting coefficient estimates $\hat{\beta}_{m,i}$

Consistent Functional PCA



(a) $\hat{\beta}_{m,i}$



(b) Data Fit

Consistent Functional PCA

- Estimate vector auto-regressive (VAR) on time-series of projection coefficients

$$\boldsymbol{\beta}_m = \mathbf{m} + \mathbf{d}\, t + \mathbf{A}\boldsymbol{\beta}_{m-1} + \varepsilon_m$$

  - $\mathbf{m}$ is a constant mean vector
  - $\mathbf{d}$ is a linear trend vector (any detrending is allowed)
  - $\mathbf{A}$ is a $K \times K$ cross-interaction matrix
  - ε_m are iid $\mathcal{N}(\mathbf{0}, \boldsymbol{\Omega})$.

- Extract "true" stochastic degrees of freedom, i.e. the **residuals**

$$\widehat{\varepsilon}_m = \boldsymbol{\beta}_m - \left(\widehat{\mathbf{m}} + \widehat{\mathbf{d}}\, t + \widehat{\mathbf{A}}\boldsymbol{\beta}_{m-1}\right)$$

Consistent Functional PCA

- Let $\mathbf{E} = (\widehat{\varepsilon}_1 \widehat{\varepsilon}_2 \dots \widehat{\varepsilon}_N)^T$ denote the matrix form of the residuals
- Define a **variance-covariance function**

$$v(\tau_1, \tau_2) \triangleq \frac{1}{N} (\mathbf{E}\phi(\tau_1))^T \mathbf{E}\phi(\tau_2)$$

- The eigen-function problem is now

$$< v, \xi > (\tau) = \lambda \, \xi(\tau)$$

where the inner product $< f, g > (\tau) \triangleq \int_{\tau_{min}}^{\tau_{max}} f(\tau, s) \, g(s) \, ds$

Consistent Functional PCA

- To solve the eigen-function problem, expand ξ onto the basis functions

$$\boldsymbol{\xi}(\tau) = \mathbf{z}\phi(\tau)$$

- Then the eigen-problem becomes

$$\frac{1}{N}(\mathbf{E}\phi(\tau))^T\mathbf{EWz} = \lambda\phi^T(\tau)\mathbf{z}$$

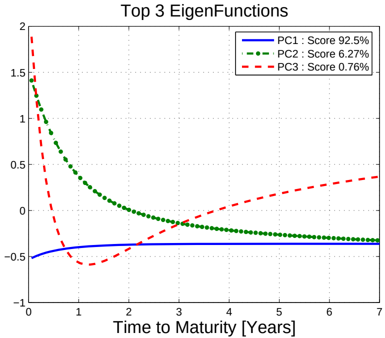
here $\mathbf{W}_{ij} \triangleq < \phi_i, \phi_j >$

- Take inner product with ϕ to find that $\mathbf{z}$ satisfy the eigen-problem

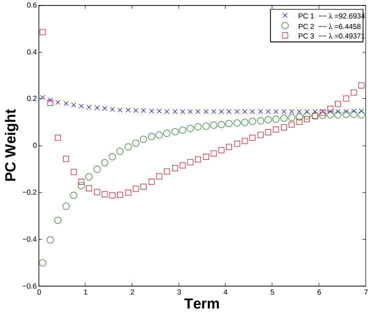
$$\frac{1}{N}\left(\mathbf{WE}^T\mathbf{E}\right)(\mathbf{Wz}) = \lambda\,(\mathbf{Wz})$$

Consistent Functional PCA

Comparison of extracted principal components Crude Oil data



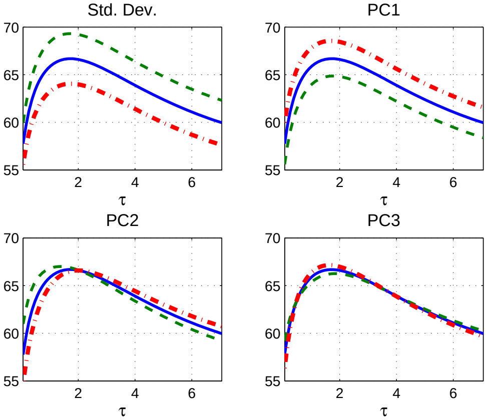
(c) Consistent FPCA



(d) Standard PCA

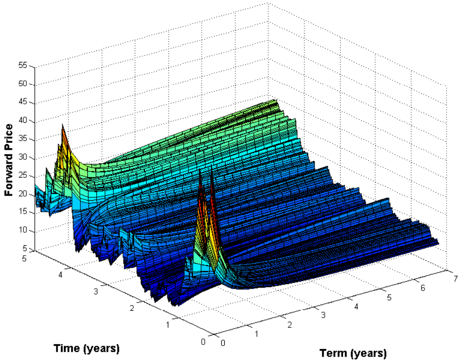
Consistent Functional PCA

Principal component perturbations on a given curve



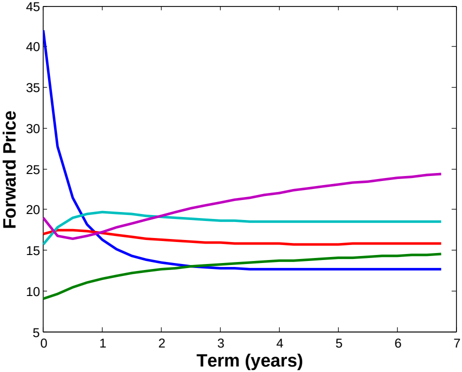
Consistent Functional PCA

Simulation forward curves using 3 principal components



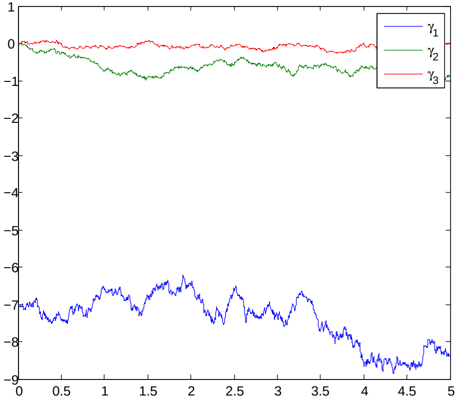
Consistent Functional PCA

Simulation forward curves using 3 principal components



Consistent Functional PCA

Simulation forward curves using 3 principal components



HJM Forward Price Models

- **Heath-Jarrow-Morton (HJM)** inspired forward price models

$$
\begin{aligned}
F_t(T) &= const. \times \exp\left\{ \int_t^T y_t(s)\, ds \right\} \\
dy_t(s) &= \mu_t(s)\, dt + \sigma_t(s)\, dW_t
\end{aligned}
$$

- This is an infinite system of SDEs (one for every maturity).
- The processes $y_t(s)$ are called **forward cost of carry**
  – analogs of instantaneous forward rates of interest
- To avoid arbitrage, the drift and volatility must satisfy the **HJM drift restrictions**

$$
\mu_t(T) = -\sigma_t(T) \int_t^T \sigma_t(s)\, ds
$$

HJM Forward Price Models

- Forward prices then evolve as

$$\frac{dF_t(T)}{F_t(T)} = \sigma_t^F(T)\, dW_t\,, \quad \sigma_t^F(T) = \int_t^T \sigma_t(s)\, ds$$

- Entire forward price curve is **matched exactly**
- With deterministic volatilities, the **forward prices are GBMs**

$$F_t(T) = F_{t_0}(T) \exp\left\{ -\frac{1}{2} \int_{t_0}^t (\sigma_u^F(T))^2\, du + \int_{t_0}^t \sigma_u^F(T)\, dW_t \right\}$$

- Can **match implied volatility** term structure with constant volatilities
- Vol smiles will require state dependent vol or stochastic vol

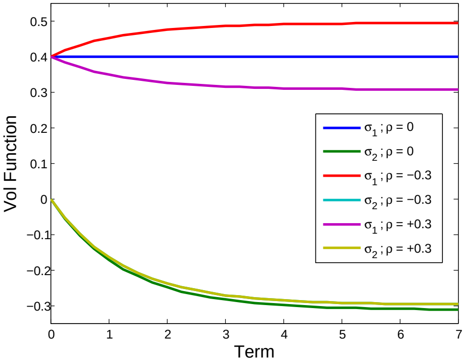
HJM Forward Price Models

- Spot price models can be recast into forward price models fairly easily
- The Schwarz (1997) stochastic convenience model corresponds to setting

$$
\begin{aligned}
\sigma_t^{F(1)}(T) &= \sigma_1 - \rho\sigma_2 \frac{1 - e^{-\kappa(T-t)}}{\kappa} \\
\sigma_t^{F(2)}(T) &= -\sigma_2\sqrt{1-\rho^2}\,\frac{1 - e^{-\kappa(T-t)}}{\kappa}
\end{aligned}
$$

HJM Forward Price Models

Volatility components in the Schwartz model



Notice that both components are affected by correlation

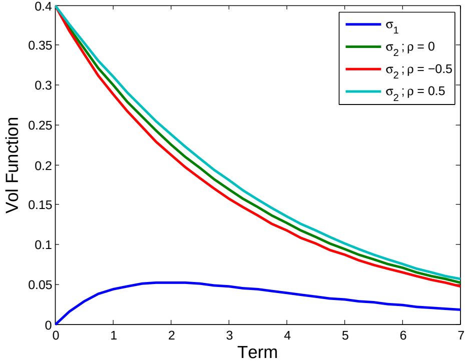
HJM Forward Price Models

- Spot price models can be recast into forward price models fairly easily
- The HJ (2007) two-factor spot model corresponds to setting

$$\begin{aligned}
\sigma_t^{F(1)}(T) &= \eta\gamma\left(e^{-\beta\tau} - e^{-\alpha\tau}\right) \\
\sigma_t^{F(2)}(T) &= \left[\sigma^2 e^{-2\beta\tau} + \eta^2\gamma^2\left(e^{-\beta\tau} - e^{-\alpha\tau}\right)^2 \right. \\
&\qquad \left. + \rho\eta\sigma\gamma\left(e^{-\beta\tau} - e^{-\alpha\tau}\right) e^{-\beta\tau}\right]^{1/2} \\
\gamma &= \frac{\beta}{\alpha - \beta}
\end{aligned}$$

HJM Forward Price Models

Volatility components in the HJ model



Notice that only the tilting component is affected by correlation

String / Market Models

- A continuum of maturity dates do not exist
- Model instead a **discrete set of forward prices** directly – not through forward cost of carry

$$\frac{dF_t(T_i)}{F_t(T_i)} = \sigma_t^F(T_i)\, dW_t$$

- Exactly fits market forward prices
- Exactly fits a given term structure of at-the-money implied volatilities
- Easy to compute call / put options on forward contracts

String / Market Models

Another way to account for the discrete nature of available maturity dates...

- Between pairs of maturity dates (T_i, T_{i+1}), define a discrete forward cost of carry

$$y_t^{(i)} \triangleq \frac{1}{T_i - T_{i-1}} \left(\frac{F_t(T_i)}{F_t(T_{i-1})} - 1 \right)$$

- Then forward prices can be recovered as

$$F_t(T_n) = S_t \prod_{k=1}^{n} (1 + (T_k - T_{k-1}) y_t^{(k)})$$

where $T_0 = t$ and recall that $F_t(t) = S_t$.

String / Market Models

- Each $y_t^{(i)}$ is martingale under a measure induced by the Radon-Nikodym derivative process

$$\left(\frac{d\mathbb{Q}^{(i)}}{d\mathbb{Q}}\right)_t \triangleq \eta_t^{(i)} = \frac{F_t(T_{i-1})}{F_0(T_{i-1})}$$

- Then assuming a diffusive model, can write

$$\frac{dy_t^{(i)}}{y_t^{(i)}} = \sigma_t^{(i)}\, dW_t^{(i)}$$

where $W_t^{(i)}$ are $\mathbb{Q}^{(i)}$-Wiener processes

String / Market Models

- Further assuming $\sigma_t^{(i)}$ are deterministic (or even just constant) provides Magrabe like formula for nearby calendar spread options

$$
\begin{aligned}
V_t &= \mathbb{E}_t^{\mathbb{Q}}[(F_T(T_i) - F_T(T_{i-1}))_+] \\
&= \frac{\mathbb{E}_t^{\mathbb{Q}^{(i)}}\left[(F_T(T_i) - F_T(T_{i-1}))_+ \left(\frac{d\mathbb{Q}}{d\mathbb{Q}^{(i)}}\right)_T\right]}{E_t^{\mathbb{Q}^{(i)}}\left[\left(\frac{d\mathbb{Q}}{d\mathbb{Q}^{(i)}}\right)_T\right]} \\
&= F_t(T_{i-1})\mathbb{E}_t^{\mathbb{Q}^{(i)}}\left[\left(\frac{F_t(T_i)}{F_t(T_{i-1})} - 1\right)_+\right]
\end{aligned}
$$