Derivatives you know and love already

$$\frac{d}{dx}\ x^n = nx^{\,n-1}$$

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}\ v - u\,\frac{dv}{dx}}{v^2}$$

$$\frac{d}{dx}\ e^{\,ax} = ae^{\,ax}$$

$$\frac{df(y(x))}{dx} = \frac{dy}{dx}\ \frac{df}{dy}$$

Partial derivative, keeping y constant in z=f(x,y)

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

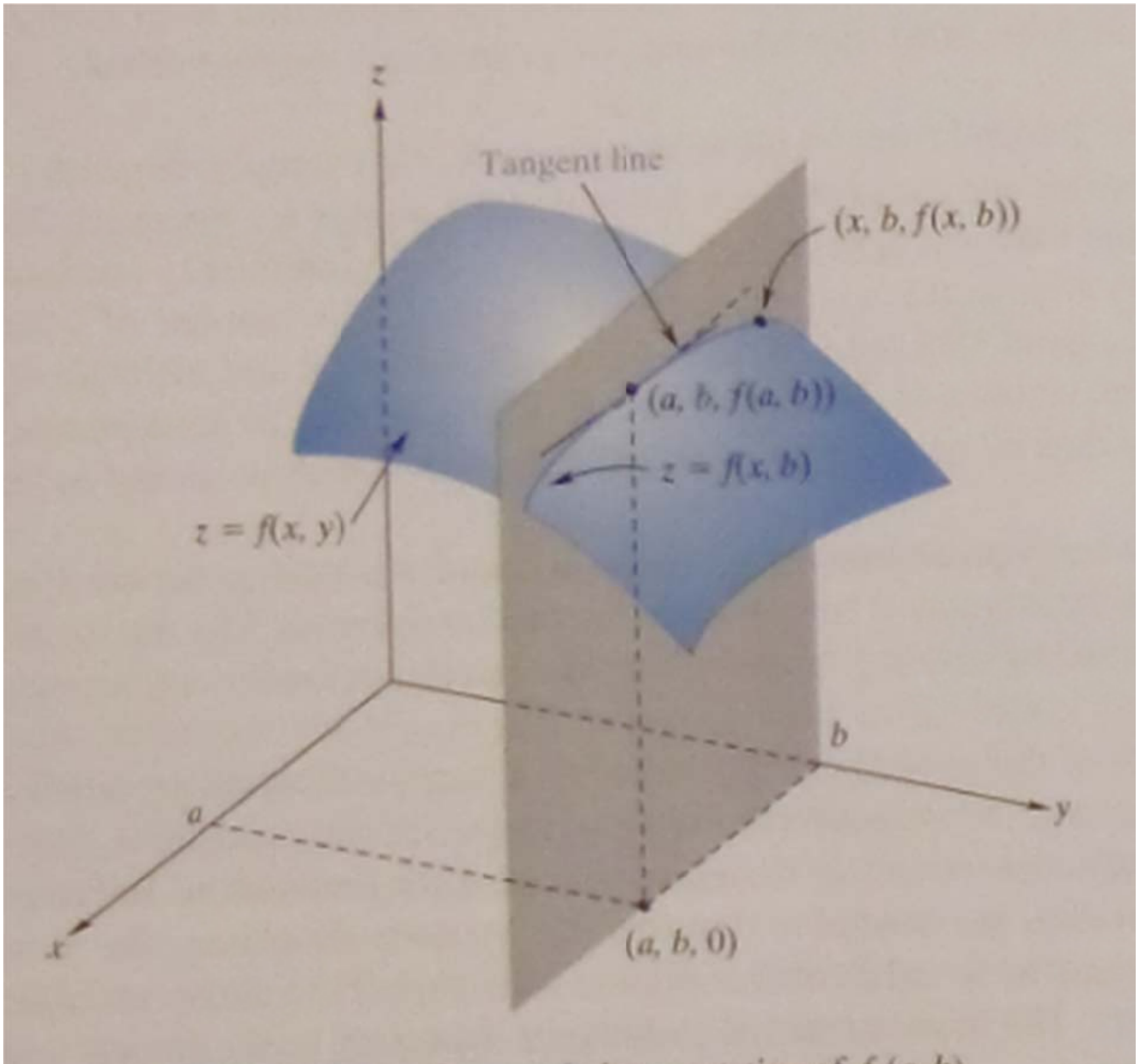


FIGURE 17.1 Geometric interpretation of $f_x(a,b)$.

Explicit Differentiation

Maybe z is an explicit function:

$z=f(x,y)$.

Then if the f consists of polynomials and other differentiable functions of x, y it's easy to calculate $f_x(x,y)$, (same as $\frac{\partial z}{\partial x}$) for constant **y.**

Example:

$z=10y + 6x + 3xy$

$\frac{\partial z}{\partial x} = 6 + 3y$ (imagine holding y constant)

$\frac{\partial z}{\partial y} = 10 + 3x$ (imagine holding x constant)

Implicit Differentiation

Maybe z it's tough to get z alone on the left hand side.

Example:

$xyz + \ x + 4y + 2z = 0$

Keep y constant, take $\frac{\partial z}{\partial x}$

$yz + +xy\frac{\partial z}{\partial x} \ +1+2\,\frac{\partial z}{\partial x} \ = 0$

$$\frac{\partial z}{\partial x} = -\frac{yz+1}{xy+2}$$

Heat Loss from Skin in cold wind

$H = (10.45 + 10\sqrt{w} - w)(33-t)$

w=wind speed (metres/second)
s=wind speed (km/hr); s=3.6w
t=temperature (Celsius)
H= heat loss of skin (kilocaries per sq metre per hr)

$H = (10.45 + 10\sqrt{(s/3.6)} - s/3.6)(33-t)$

$= (10.45 + 5.27\sqrt{s} - 0.278s)(33-t)$

Example Heat01
How many times more heat loss on Monday (s=40 km/h, t=-8) than last week (s= 10 km/h, t=0)?

H(last week)=H(s,t)=H(10,0)
$= (10.45 + 5.27\sqrt{10} - 0.278*10)(33-(-8)) = 803.0617$

H(Monday)=H(40,-8) =1339.077

Monday heat loss was 1339/803=1.67 times as fast.

Example Heat02

Starting at s=20 km/hr and t= -10C, what is the proportional increase in heat loss per extra km/hr?

$H= (10.45 + 5.27\sqrt{s} - 0.278s)(33-t)$

$H(20, -10)=1223.7$

$\frac{\partial H}{\partial s}=(0.5*5.27/\sqrt{s} - 0.278)(33-t)$

$= (0.5*5.27/\sqrt{20} - 0.278)(33+10)=13.382$

Proportional % increase for extra 1 km/hr=13.382/1223.7=1.1%

Question: Is this H formula believable?

$H = (10.45 + 5.27\sqrt{s} - 0.278s)(33-t)$

Note that $\frac{dH}{ds} = (0.5*5.27/\sqrt{s} - 0.278)(33-t)$ takes the value

$0 = \frac{dH}{ds}$ at $(s,t) = ((0.5*5.27/0.278)^2, t) = (90, t)$

$0 > \frac{dH}{ds}$ for $s > 90$

So above 90 km/hour, (25 metres/sec) higher wind speed reduces the heat loss? Clearly wrong, so the formula is probably an approximation which works only for low wind speeds.

MATA33 Final W09 Q5

5. If $w = \sqrt[3]{rs}\, e^{5+r}$ then $\dfrac{\partial w}{\partial s}$ equals $\quad w = r^{1/3} s^{1/3} e^{5+r}$

(a) $\dfrac{w}{s}$ (b) $\dfrac{3w}{s}$ (c) $\dfrac{w}{3s}$ (d) $\dfrac{sw}{3}$ (e) none of (a) - (d)

$$\frac{\partial w}{\partial s} = r^{1/3}\,\frac{s^{-2/3}}{3}\,e^{5+r}$$

$$= \frac{r^{1/3} s^{1/3} e^{5+r}}{3s}$$

$$= \frac{w}{3s}$$

3. Assume the equation $\overbrace{2z^2 + 2x^2z^3 = xy}^{(*)}$ defines z implicitly as a function of independent variables x and y. Find the value of z_x at the point (x, y, z) where $y = 4$ and $z = 1$.

[7 points]

Partial differentiate w.r.t. x :

$$4zz_x + 4xz^3 + 6x^2z^2z_x = y$$

$$z_x(4z + 6x^2z^2) = y - 4xz^3$$

$$z_x = \frac{y - 4xz^3}{4z + 6x^2z^2} \qquad z_x(1, 4, 1) = \frac{0}{10}$$

When $y = 4, z = 1$ in $(*)$, $\qquad = 0$

$$2 + 2x^2 = 4x$$

$$2x^2 - 4x + 2 = 0$$

$$2(x-1)^2 = 0 \rightarrow x = 1$$

$\boxed{\text{Answer} = 0}$

Marginal Costs:

How much extra does it cost Huang Industries to manufacture one more car, n+1, rather than n, while continuing to make m motorcycles? 'Marginal', depending on interpretation, doesn't usually include the fact that the extra car makes building a second factory a little more desirable. And it probably doesn't include paying the company president any more. Cost C(n, m) is unlikely to be exactly an addition of kg of materials and hours: what about overcrowding, quantity discounts on materials, overtime shifts?

We define the derivate - $\frac{\partial C}{\partial c}$as the marginal cost. Economic theory says that under 'perfect competition' the price charged is driven down to the marginal cost. McDonalds will undercut Burger King by a couple of cents if it brings a profitable customer in, and then BK reduces its prices etc. Or the Toronto Star lets us view their sites for the cost of seeing a 15 second candy ad. (So we click on the Globe site instead, and watch their 10 second candy ad. But how to pay the journalists?? And for most of us the ad is wasted time – we would prefer to pay one cent, but how to do that?).

MATA33 Final Exam W09?

7. In all of this question $C = \dfrac{xy}{5x + 3y}$ is the manufacturing cost function (in millions of dollars) where $x, y > 0$ are the number of hundreds of units of two products, P and Q, respectively.

(a) Find and simplify the marginal cost functions. [4 + 4 points]

$$C_x = \frac{y(5x+3y) - 5xy}{(5x+3y)^2} = \frac{3y^2}{(5x+3y)^2}$$

$$C_y = \frac{x(5x+3y) - 3xy}{(5x+3y)^2} = \frac{5x^2}{(5x+3y)^2}$$

(b) Find and interpret the meaning of $\dfrac{\partial C}{\partial x}(3,1)$ [3 points]

$$\frac{\partial C}{\partial x}(3,1) = C_x(3,1) = \frac{3}{(18)^2} = \frac{1}{108}$$

Interpretation $\dfrac{1}{108} \approx C(4,1) - C(3,1)$

i.e. Cost to manufacture the 4^{th} batch of 100 units of P while making 100 units of Q is $\approx 1/108$

(c) Find the number of units of P and Q ...

Demand functions

Quantity sold q_A Android phones and q_B Blackberries with prices p_A and p_B.

$q_A = f(p_A, p_B)$

$q_B = g(p_A, p_B)$

Marginal demand functions are partial derivatives with respect to one of the prices.

Almost always, we buy less of something expensive:

$$\frac{\partial q_A}{\partial p_A} < 0$$

$$\frac{\partial q_B}{\partial p_B} < 0$$

(``Almost always` but e.g. lager beer marketers in Britain found that charging an extra few pennies actually increased purchases – probably people trying to impress their mates!)

Demand Functions: Competitive Products (Substitutes)

Androids and Blackberries

Butter and margarine
Tea and coffee
Wine and beer

$\frac{\partial q_A}{\partial p_B} > 0$ (switch to cheap tea from expensive coffee)

$$\frac{\partial q_B}{\partial p_A} > 0$$

Example:

$q_A = 1000 \frac{p_B}{p_A}$ (hamburgers vs subs; each about \$5)

$\frac{\partial q_A}{\partial p_B} = 1000 \frac{1}{p_A} > 0$ (substitutes)

Demand Functions: Complementary Products

Cars and gasoline

BluRay discs and BluRay players

Itunes tracks and IPods

Pirated tracks and IPods (Apple`s route to success maybe)

$\frac{\partial q_A}{\partial p_A} < 0$ (buy ebook reader if cheap or ebooks available)

$\frac{\partial q_B}{\partial p_B} < 0$

Example:

$q_A = 10{,}000 \frac{1}{p_B \; p_A}$ ($2 sausages and $1 buns)

$\frac{\partial q_A}{\partial p_B} = -10{,}000 \frac{1}{p_B{}^2 \; p_A} < 0$ (complements)

Haeussler Example 1 on Marginal Costs

A company manufactures two types of skis, the Lightning and the Alpine models. Suppose the joint-cost function for producing x pairs of the Lightning model and y pairs of the Alpine model per week is

$$c = f\left(x, y\right) = 0.07x^2 + 75x + 85y + 6000$$

where c is expressed in dollars. Determine the marginal costs $\partial c/\partial x$ and $\partial c/\partial y$ when x = 100 and y = 50, and interpret the results.

Solution: The marginal costs are

$$\frac{\partial c}{\partial x}(at100, 50) = 0.14x + 75 = \$89 \quad \text{and}$$

$$\frac{\partial c}{\partial y}(at100, 50) = \$85$$

Haeussller Example 3 on Marginal Productivity

A manufacturer of a popular toy has determined that the production function is P = √(lk), where l is the number of labor-hours per week and k is the capital (expressed in hundreds of dollars per week) required for a weekly production of P gross of the toy. (One gross is 144 units.) Determine the marginal productivity functions, and evaluate them when l = 400 and k = 16. Interpret the results.

Solution

$$\frac{d}{dx} x^n = n x^{n-1} \qquad \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx} v - u\frac{dv}{dx}}{v^2} \qquad \frac{df(y(x))}{dx} = \frac{dy}{dx}\frac{df}{dy}$$

$$P = (lk)^{1/2}$$

$$\frac{\partial P}{\partial l} = \frac{1}{2}(lk)^{-1/2} k = \frac{k}{2\sqrt{lk}} \quad \text{and} \quad \frac{\partial P}{\partial k} = \frac{l}{2\sqrt{lk}}$$

$$\left.\frac{\partial P}{\partial l}\right|_{l=400,k=16} = \frac{1}{10} \quad \text{and} \quad \left.\frac{\partial P}{\partial k}\right|_{l=400,k=16} = \frac{5}{2}$$

Haeussler 17.3: Implicit Partial Differentiation

A function z defined **implicitly:**

If $$\frac{xz^2}{x+y} + y^2 = 0$$

Evaluate $\dfrac{\partial z}{\partial x}$ when x=-1, y=2 and z=2.

Solution:

$$\frac{d}{dx} x^n = nx^{n-1} \qquad \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx} v - u\frac{dv}{dx}}{v^2} \qquad \frac{df(y(x))}{dx} = \frac{dy}{dx}\frac{df}{dy}$$

$$\frac{\partial}{\partial x}\left(\frac{xz^2}{x+y}\right) + \frac{\partial}{\partial x}\left(y^2\right) = \frac{\partial}{\partial x}(0)$$

$$\left(\frac{(z^2(x+y) + 2xz\frac{\partial z}{\partial x}(x+y) - xz^2}{(x+y)^2}\right) + \frac{\partial}{\partial x}\left(y^2\right) = \frac{\partial}{\partial x}(0)$$

$$2xz\left(x+y\right)\frac{\partial z}{\partial x} + z^2\left(x+y\right) - xz^2 = 0$$

$$\frac{\partial z}{\partial x} = -\frac{yz}{2x\left(x+y\right)} \qquad z \neq 0$$

$$\left.\frac{\partial z}{\partial x}\right|_{(-1,2,2)} = 2$$

MATA33 W09 TT2 Q01

1. If $f(x,y) = 5x^2 ln(x^2 - 3y)$ then $f_x(2,1)$ equals

(a) 20 (b) 80 (c) 81 (d) 101 (e) none of (a) - (d)

$$f_x(x,y) = 10x\ln(x^2-3y) + \frac{5x^2(2x)}{x^2-3y}$$

$$f_x(2,1) = 20\ln(1) + \frac{80}{1} = \boxed{80}$$

$\frac{d}{dx}x^n = nx^{n-1}$ $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}$ $\frac{df(y(x))}{dx} = \frac{dy}{dx}\frac{df}{dy}$

MATA33 W09 TT2 Q02

2. The joint demand functions for products A and B are $\alpha(x, y) = 80 - 2x + e^{-y} - y^2$ and $\beta(x, y) = 140 + (4x)^{-1} - 7y$ respectively, and x and y are the unit prices for A and B, respectively. We may then conclude that A and B are

(a) complementary (b) competitive (c) both (a) and (b) (d) neither (a) nor (b)

$$\frac{\partial \alpha}{\partial y} = -exp(-y) - 2y \ < 0$$

$$\frac{\partial \beta}{\partial x} = -\frac{1}{4x^{\,2}} \ < 0$$

Note that to show complementary we need to check both derivatives.

MATA33 W09 TT2 Q07a

7. In all of this question $C = \dfrac{xy}{5x+3y}$ is the manufacturing cost function (in millions of dollars) where $x, y > 0$ are the number of hundreds of units of two products, P and Q, respectively.

   (a) Find and simplify the marginal cost functions. [4 + 4 points]

$$\frac{d}{dx} x^n = nx^{n-1} \qquad \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2} \qquad \frac{df(y(x))}{dx} = \frac{dy}{dx}\frac{df}{dy}$$

$$C_x = \frac{y(5x+3y) - 5xy}{(5x+3y)^2} = \frac{3y^2}{(5x+3y)^2}$$

$$C_y = \frac{x(5x+3y) - 3xy}{(5x+3y)^2} = \frac{5x^2}{(5x+3y)^2}$$

MATA33 W09 TT2 Q07b

7. In all of this question $C = \dfrac{xy}{5x+3y}$ is the manufacturing cost function (in millions of dollars) where $x, y > 0$ are the number of hundreds of units of two products, P and Q, respectively.

(b) Find and interpret the meaning of $\dfrac{\partial C}{\partial x}(3,1)$

$$\frac{\partial C}{\partial x}(3,1) = C_x(3,1) = \frac{3}{(18)^2} = \frac{1}{108}$$

Interpretation $\dfrac{1}{108} \approx C(4,1) - C(3,1)$

ie Cost to manufacture the 4th batch of 100 units of P while making 100 units of Q is $\approx 1/108$

MATA33 W09 TT2 Q07c

7. In all of this question $C = \dfrac{xy}{5x+3y}$ is the manufacturing cost function (in millions of dollars) where $x, y > 0$ are the number of hundreds of units of two products, P and Q, respectively.

(c) Find the number of units of P and Q manufactured under the assumptions that (i) the total number manufactured is 1,000 units and (ii) the marginal cost functions are equal. Round your answers to the nearest unit.

[6 points]

(i) $\Rightarrow x + y = 10$ (x, y are in units of 100's)

(ii) $C_x = C_y \Rightarrow \dfrac{3y^2}{(5x+3y)^2} = \dfrac{5x^2}{(5x+3y)^2}$

$\therefore y = \sqrt{\dfrac{5}{3}}\, x$ as $x, y > 0$

(extra answer space for Part (c) is on the next page)

By (i), $x\left(1+\sqrt{\dfrac{5}{3}}\right) = 10$

$\therefore x = \dfrac{10}{1+\sqrt{\frac{5}{3}}} \approx 4.3649$

$y \approx 5.635$

$\boxed{\therefore \text{ 436 of } P \text{ and 564 of } Q.}$

MATA33 S09 TT2 Q04

4. Assume the equation $\overbrace{e^{xz} = xyz}^{}$ defines the variable z implicitly as a function of the two independent variables x and y.

Evaluate z_y at the point (x, y, z) where $x = 1$ and $z = -1$. A complete answer requires that you find the value of y for the point (x, y, z). [10 points]

To find y when $x = 1$ and $z = -1$ sub in $\circledast$

$$e^{-1} = -y \Rightarrow y = -\frac{1}{e}$$

$\therefore$ the point (x, y, z) is $\left(1, -\frac{1}{e}, -1\right)$

MATA33 S09 TT2 Q04 (cont)

4. Assume the equation $e^{xz} = xyz$ defines the variable z implicitly as a function of the two independent variables x and y.

   Evaluate z_y at the point (x, y, z) where $x = 1$ and $z = -1$. A complete answer requires that you find the value of y for the point (x, y, z). [10 points]

$$\frac{d}{dx}\, x^n = n x^{n-1} \qquad \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}\, v - u\frac{dv}{dx}}{v^2} \qquad \frac{df(y(x))}{dx} = \frac{dy}{dx}\frac{df}{dy}$$

Now we get z_y :

$$\frac{\partial}{\partial y}\left(e^{xz}\right) = \frac{\partial}{\partial y}(xyz)$$

$$e^{xz} \cdot z_y = xz + xy\, z_y$$

$$z_y\left(e^{xz} - xy\right) = xz$$

$$\therefore \quad z_y = \frac{xz}{e^{xz} - xy}$$

$$z_y\left(1, -\tfrac{1}{e}, -1\right) = \frac{-1}{e^{-1} + e^{-1}} = \frac{-1}{\left(\frac{2}{e}\right)} = -\frac{e}{2}$$

MATA33 S09 TT2 Q05a

5. In all of this question assume $c(x,y) = x\sqrt{y}\sqrt{x+y}$ is a joint cost function in dollars where $x, y > 0$ are the numbers of units of products X and Y respectively.

(a) Find the marginal cost functions. Express your answers using radicals, not exponents.

$$\frac{d}{dx}x^n = nx^{n-1} \qquad \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2} \qquad \frac{df(y(x))}{dx} = \frac{dy}{dx}\frac{df}{dy}$$

$$c_x(x,y) = \sqrt{y}\sqrt{x+y} + \frac{x\sqrt{y}}{2\sqrt{x+y}}$$

$$c_y(x,y) = \frac{x\sqrt{x+y}}{2\sqrt{y}} + \frac{x\sqrt{y}}{2\sqrt{x+y}}$$

(b) State the mathematical approximation involving the cost function for units $x = 36$, $y = 64$, $y = 65$, and one of the marginal cost functions in part (a). Evaluate these functions and comment on the accuracy of this approximation. [6 points]

Desired approximation statement is:

$$c(36,65) - c(36,64) \approx c_y(36,64) \quad ☺$$

$$c(36,65) = 36\sqrt{65}\sqrt{101} \approx 2{,}916.8888$$

$$c(36,64) = 36(8)(10) = 2880$$

$$c_y(36,64) = \frac{36(10)}{16} + \frac{36(8)}{20}$$

$$= 22.5 + 14.4 = 36.9$$

$$c(36,65) - c(36,64) \approx 36.8888$$

The approximation in ☺ is extremely accurate. The difference between left & right sides in ☺ is about .0112

(c) Verify that if $y > \frac{x}{2}$ then $c_x(x, y) > c_y(x, y)$ [6 points]

Assume $y > \frac{x}{2} > 0$ ($\because x, y > 0$ as given)

$$c_x(x,y) - c_y(x,y)$$

$$= \sqrt{y}\sqrt{x+y} + \frac{x\sqrt{y}}{2\sqrt{x+y}} - \frac{x\sqrt{x+y}}{2\sqrt{y}} - \frac{x\sqrt{y}}{2\sqrt{x+y}}$$

$$= \sqrt{y}\sqrt{x+y} - \frac{x\sqrt{x+y}}{2\sqrt{y}}$$

$$= \underbrace{\frac{\sqrt{x+y}}{\sqrt{y}}}_{>0} \underbrace{\left(y - \frac{x}{2}\right)}_{>0} > 0 \quad \left(\text{as } y > \frac{x}{2}\right)$$

$\therefore c_x(x,y) > c_y(x,y)$ as required.

Haeussler 17.4: Higher-Order Partial Derivatives

- We obtain second-order partial derivatives of f as

$$f_{xx} \text{ means } (f_x)_x \quad \text{and } f_{xy} \text{ means } (f_x)_y$$

$$f_{yx} \text{ means } (f_y)_x \quad \text{and } f_{yy} \text{ means } (f_y)_y$$

Example 1 – Second-Order Partial Derivatives

Find the four second-order partial derivatives of $f(x,y)=x^2y+x^2y^2$.

Solution

$$f_x(x,y)=2xy+2xy^2$$

$$f_{xx}(x,y)=2y+2y^2 \quad \text{and} \quad f_{xy}(x,y)=2x+4xy$$

$$f_y(x,y)=x^2+2x^2y$$

$$f_{yy}(x,y)=2x^2 \quad \text{and} \quad f_{yx}(x,y)=2x+4xy$$

Example 3 – Second-Order Partial Derivative of an Implicit Function

Determine $\frac{\partial^2 z}{\partial x^2}$ if $z^2 = xy$.

Solution: By implicit differentiation,

$$\frac{\partial}{\partial x}\left(z^2\right) = \frac{\partial}{\partial x}\left(xy\right) \Rightarrow 2z\frac{\partial z}{\partial x} = y \Rightarrow \frac{\partial z}{\partial x} = \frac{y}{2z} \qquad z \neq 0$$

$$\frac{d}{dx}x^n = nx^{n-1} \qquad \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2} \qquad \frac{df(y(x))}{dx} = \frac{dy}{dx}\frac{df}{dy}$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{1}{2}yz^{-1}\right) \Rightarrow \frac{\partial^2 z}{\partial x^2} = -\frac{1}{2}yz^{-2}\frac{\partial z}{\partial x}$$

Substituting $\frac{\partial z}{\partial x} = \frac{y}{2z}$,

$$\frac{\partial^2 z}{\partial x^2} = -\frac{1}{2}yz^{-2}\left(\frac{y}{2z}\right) = -\frac{y^2}{4z^3} \qquad z \neq 0$$

Haeussler 17.5: Chain Rule

- If f, x, and y have continuous partial derivatives, then z is a function of r and s, and

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

and $$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

Example 1 – Rate of Change of Cost

For a manufacturer of cameras and film, the total cost c of producing q_C cameras and q_F units of film is given by

$$c = 30q_C + 0.015q_C q_F + q_F + 900$$

The demand functions for the cameras and film are given by

$$q_C = \frac{9000}{p_C\sqrt{p_F}} \quad \text{and} \quad q_F = 2000 - p_C - 400p_F$$

where p_C is the price per camera and p_F is the price per unit of film. Find the rate of change of total cost with respect to the camera price when $p_C = 50$ & $p_F = 2$.

Solution (Use the chain rule)

$$\frac{\partial c}{\partial p_C} = \frac{\partial c}{\partial q_C}\frac{\partial q_C}{\partial p_C} + \frac{\partial c}{\partial q_F}\frac{\partial q_F}{\partial p_C} = \left(30 + 0.015q_F\right)\left[\frac{-9000}{p_C^2\sqrt{p_F}}\right] + \left(0.015q_C + 1\right)\left(-1\right)$$

$$\left.\frac{\partial c}{\partial p_C}\right|_{p_C=50,\, p_F=2} \approx -123.2$$

Example 3a – Chain Rule

Determine $\partial y/\partial r$ if $y = x^2 \ln\left(x^4+6\right)$ and $x = \left(r+3s\right)^6$.

Solution (chain rule)

$$\frac{\partial y}{\partial x} = \left[\frac{4x^5}{x^4+6} + 2x\ln\left(x^4+6\right)\right] = 2x\left[\frac{2x^4}{x^4+6} + \ln\left(x^4+6\right)\right]$$

$$\frac{\partial y}{\partial r} = \frac{\partial y}{\partial x}\frac{\partial x}{\partial r} = 12x\left(r+3s\right)^5\left[\frac{2x^4}{x^4+6} + \ln\left(x^4+6\right)\right]$$

Example 3b – Chain Rule

Given that $z = e^{xy}$, $x = r - 4s$, and $y = r - s$, find $\partial z/\partial r$ in terms of r and s.

Solution (chain rule)

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \left(x + y\right)e^{xy}$$

$$= \left(2r - 5s\right)e^{r^2 - 5rs + 4s^2}$$

MATA33 W09 TT2 Q3

3. Assume the equation $\overbrace{2z^2 + 2x^2z^3 = xy}$ defines z implicitly as a function of independent variables x and y. Find the value of z_x at the point (x, y, z) where $y = 4$ and $z = 1$.

[7 points]

Partial differentiate w.r.t. x :

$$4zz_x + 4xz^3 + 6x^2z^2z_x = y$$

$$z_x(4z + 6x^2z^2) = y - 4xz^3$$

$$z_x = \frac{y - 4xz^3}{4z + 6x^2z^2}$$

When $y = 4, z = 1$ in (*),

$$2 + 2x^2 = 4x$$

$$2x^2 - 4x + 2 = 0$$

$$2(x-1)^2 = 0 \rightarrow x = 1$$

$$z_x(1, 4, 1) = \frac{0}{10}$$

$$= 0$$

$$\boxed{\text{Answer} = 0}$$

MATA33 W09 TT2 Q4

4. Let $f(x,y) = e^{xy}$ and let $F(x,y) = f_y(x,y) - f_{xy}(x,y)$ Find the function $y = g(x)$ such that $F(x, g(x)) = 0$ for all x in the domain of g.

[8 points]

$$f_y(x,y) = e^{xy} \cdot x$$

$$f_{xy}(x,y) = f_{yx}(x,y) = e^{xy}xy + e^{xy}$$

$$F(x,y) = e^{xy}x - e^{xy}xy - e^{xy}$$

$$= e^{xy}(x - xy - 1)$$

$$F(x,y) = 0 \iff x - xy - 1 = 0$$

$$\iff y = \frac{x-1}{x}$$

$$\boxed{\therefore \; g(x) = \frac{x-1}{x}, \; x \neq 0}$$

Haeussler 17.6: Extrema for Functions of two variables

Relative maximum at the point (a, b) is shown as

$$f\left(a,b\right) \geq f\left(x,y\right)$$

RULE 1

Find relative maximum or minimum when

$$\begin{cases} f_x\left(x,y\right)=0 \\ f_y\left(x,y\right)=0 \end{cases}$$

RULE 2 Second-Derivative Test for Functions of Two Variables

Let D be the function defined by

$$D(x,y)=f_{xx}(x,y)\,f_{yy}(x,y)-\left(f_{xy}(x,y)\right)^2.$$

(call this the determinant of the Hessian matrix if you like. If you don't like, that's fine too)

1. If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, relative maximum at (a, b);
2. If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, relative minimum at (a, b);
3. If $D(a, b) < 0$, then f has a saddle point at (a, b);
4. If $D(a, b) = 0$, no conclusion.

Example 1: Critical Points

a. $f(x,y) = 2x^2 + y^2 - 2xy + 5x - 3y + 1$

Solution: Find critical points:

$$f_x(x,y) = 4x - 2y + 5 = 0$$

and

$$f_y(x,y) = -2x + 2y - 3 = 0,$$

we solve the system and get

$$\begin{cases} x = -1 \\ y = \frac{1}{2} \end{cases}$$

b. $f(l,k)=l^3+k^3-lk$

Solution: Find critical points:

$$f_l(l,k)=3l^2-k=0$$
$$f_k(l,k)=3k^2-l=0,$$

we solve the system and get

$$\begin{cases} l=0 \\ k=0 \end{cases}$$

$$\begin{cases} l=\frac{1}{3} \\ k=\frac{1}{3} \end{cases}.$$

c. $f(x,y,z)=2x^2+xy+y^2+100-z(x+y-100)$

Solution: Find critical points:

$$f_x(x,y,z)=4x+y-z=0$$
$$f_y(x,y,z)=x+2y-z=0$$
$$f_z(x,y,z)=-x-y+100=0$$

we solve the system and get

$$\begin{cases} x=25 \\ y=75 \\ z=175 \end{cases}$$

MATA33 SUM09 Final

2. Which of the following are critical points of $f(x, y) = x^3 - 3xy - y^3$?

(a) $(0, 0)$ (b) $(1, -1)$ (c) $(-1, 1)$ (d) $(1, 1)$

(e) (a) and (b) (f) (a) and (c) (g) (a) and (d)

MATA33 SUM09 Final

4. If $z = (xy)^{1/2} e^{xy}$ then $\dfrac{\partial z}{\partial y}$ equals

(a) $\dfrac{z}{2} + xz$ (b) $\dfrac{yz}{2} + xz$ (c) $\dfrac{z}{2y} + xz$ (d) $\dfrac{z}{2y} + z$ (e) none of (a) - (d)

MATA33 SUM09 Final

5. Assume the equation $w^2 = 12x - 15y$ defines w implicitly as a function of independent variables x and y. The value of $w_{xx}(2, 1, -3)$ is

(a) $-4/3$ (b) $4/3$ (c) $-3/4$ (d) $3/4$ (e) none of (a) - (d)

Example 3 – Applying the Second-Deriv Test

Examine f(x,y) = $x^3 + y^3 - xy$ *for relative maxima or minima by using the second derivative test.*

Solution: We find critical points,

$$f_x(x,y) = 3x^2 - y = 0$$
$$f_y(x,y) = 3y^2 - x = 0$$

which gives (0, 0) and (1/3, 1/3).

Calculate our test function D:

$$f_{xx}(x,y) = 6x$$
$$f_{yy}(x,y) = 6y$$
$$f_{xy}(x,y) = -1$$

$$D(x,y) = f_{xx}(x,y) f_{yy}(x,y) - \left(f_{xy}(x,y)\right)^2.$$
$$D(x,y) = (6x)(6y) - (-1)^2 = 36xy - 1$$

D(0,0)<0, hence no relative extremum at (0,0)

D(1/3, 1/3)>0, f_{xx}(1/3,1/3)>0, so minimum at(1/3,1/3)

Value of the function is:

$$f\left(\tfrac{1}{3},\tfrac{1}{3}\right)=\left(\tfrac{1}{3}\right)^3+\left(\tfrac{1}{3}\right)^3-\left(\tfrac{1}{3}\right)\left(\tfrac{1}{3}\right)=-\tfrac{1}{27}$$

Example 5 – Finding Relative Extrema

Examine $f(x, y) = x^4 + (x - y)^4$ for relative extrema.

Solution: We find critical points at (0,0) through

$$f_x(x,y)=4x^3+4(x-y)^3=0$$

$$f_y(x,y)=-4(x-y)^3=0$$

$$f_{xx}(x,y)=12x^2+12(x-y)^2=0$$

$$f_{yy}(x,y)=12(x-y)^2$$

$$f_{xy}(x,y)=0$$

D(0, 0) = 0 ➔ no information.

f has a relative (and absolute) minimum at (0, 0).

Example 7 – Profit Maximization

A candy company produces two types of candy, A and B, for which the average costs of production are constant at \$2 and \$3 per pound, respectively. The quantities q_A, q_B (in pounds) of A and B that can be sold each week are given by the joint-demand functions

$$q_A = 400(p_B - p_A)$$
$$q_B = 400(9 + p_A - 2p_B)$$

where p_A and p_B are the selling prices (in dollars per pound) of A and B, respectively. Determine the selling prices that will maximize the company's profit P.

Solution:

$$q_A = 400\left(p_B - p_A\right)$$

$$q_B = 400\left(9 + p_A - 2p_B\right)$$

$$P = \begin{pmatrix}\text{profit}\\ \text{per pound}\\ \text{of A}\end{pmatrix}\begin{pmatrix}\text{pounds}\\ \text{of A}\\ \text{sold}\end{pmatrix} + \begin{pmatrix}\text{profit}\\ \text{per pound}\\ \text{of B}\end{pmatrix}\begin{pmatrix}\text{pounds}\\ \text{of B}\\ \text{sold}\end{pmatrix}$$

The profits per pound are (p_A – 2) and (p_B – 3),

$$P = \left(p_A - 2\right)q_A + \left(p_B - 3\right)q_B$$

$$\frac{\partial P}{\partial p_A} = -2p_A + 2p_B - 1 = 0 \quad \text{and} \quad \frac{\partial P}{\partial p_B} = 2p_A - 4p_B + 13 = 0$$

The solution is p_A = 5.5 and p_B = 6.

$$\frac{\partial^2 P}{\partial p_A^2} = -800 \qquad \frac{\partial^2 P}{\partial p_B^2} = -1600 \qquad \frac{\partial^2 P}{\partial p_B \partial p_A} = 800$$

$$D\left(5.5,\ 6\right) > 0$$

Since $\partial^2 P/\partial p^2$A< 0, we indeed have a maximum.

MATA33 SUM09 Final (New)

1. Let $f(x,y) = x^2 + 2y^2 - x^2y$ Find the critical point(s) of f. For each one, use the 2^{nd}–derivative test to determine whether it corresponds to a relative maximum, minimum, or a saddle point. [11 points]

Critical points:

$$0 = \frac{\partial f}{\partial x} = 2x - 2xy = 2x(1-y)$$

$$0 = \frac{\partial f}{\partial y} = 4y - x^2$$

Critical points are (x=0, y=0) , (x=2, y=1), (x=-2,y=1)

$$\frac{\partial^2 f}{\partial x^2} = 2 - 2y$$

$$\frac{\partial^2 f}{\partial y^2} = 4$$

$$\frac{\partial^2 f}{\partial x \partial y} = -2x$$

Critical point (x=0, y=0):

$$D(x,y) = f_{xx}(x,y) f_{yy}(x,y) - \left(f_{xy}(x,y)\right)^2. \quad = 8 > 0$$

$\frac{\partial^2 f}{\partial y^2} = 4 > 0$ so (0,0) is a relative minimum.

Critical point (x=2, y=1):
$D(2, 1) = 0*4 - (-4)^2 = -16 < 0$
so (2,1) is a saddle

Critical point (x=-2, y=1):
$D(-2, 1) = 0 - (4)^2 = -16$
so (-2,1) is a saddle

(note that f(x,y) only involves x through a x^2 term so it must be symmetric about the vertical axis)

MATA33 SUM09 Final (NEW)

6. In all of this question let $f(x, y) = xy + \frac{a^3}{x} + \frac{b^3}{y}$ where a and b are arbitrary non-zero real constants of opposite signs.

(a) Find the unique critical point of f. [6 points]

$\frac{\partial f}{\partial x}$=y-$\frac{a^3}{x^2}$=0

$\frac{\partial f}{\partial y}$=x-$\frac{b^3}{y^2}$=0

x-$\frac{b^3}{a^6}$ x^4=0:

x=$\frac{a^2}{b}$

y=$\frac{b^2}{a}$

f=3ab at this point

(b) Use your answer to part (a) and the 2^{nd}–derivative test to verify that f has a local maximum value of $3ab$ [8 points]

$\frac{\partial^2 f}{\partial x^2}$=$\frac{2a^3}{x^3}$ = $\frac{2\ b^3}{a^3}$ <0 at critical

$\frac{\partial^2 f}{\partial y^2}$=2$\frac{b^3}{y^3}$ = 2$\frac{a^3}{b^3}$ <0at critical

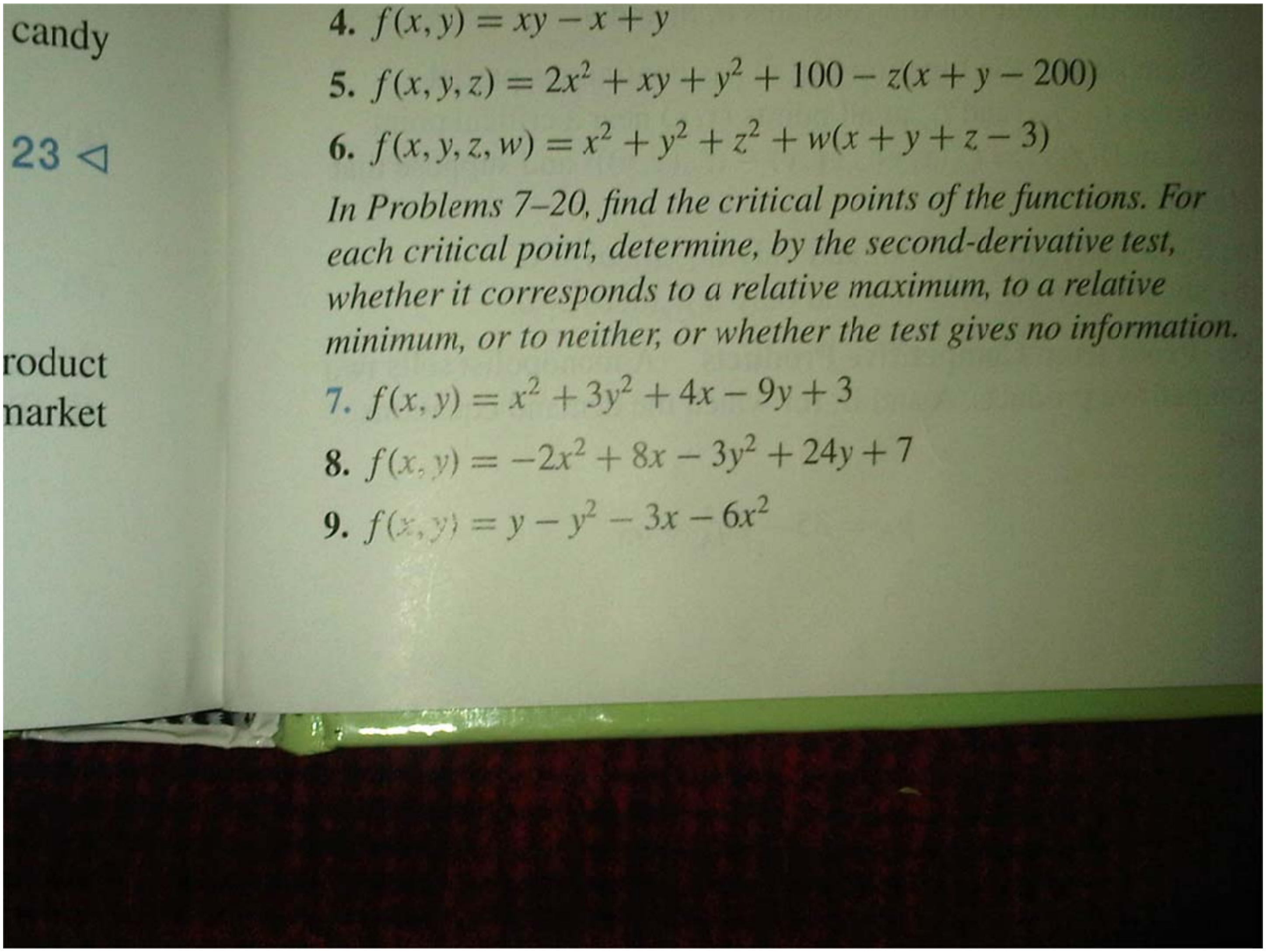
$$\frac{\partial^2 f}{\partial x\, \partial y} = 1$$

$$D(x,y) = f_{xx}(x,y)\, f_{yy}(x,y) - \left(f_{xy}(x,y)\right)^2 .$$

Critical point has D=4-1=3>0 so it's a relative minimum

(c) Show (mathematically) why f does not have an absolute maximum. [2 points]

The relative minimum has f=3ab<0 since a and b are of opposite sign. f(0,0)=0 is larger



candy

23 ◁

roduct
narket

4. $f(x, y) = xy - x + y$

5. $f(x, y, z) = 2x^2 + xy + y^2 + 100 - z(x + y - 200)$

6. $f(x, y, z, w) = x^2 + y^2 + z^2 + w(x + y + z - 3)$

In Problems 7–20, find the critical points of the functions. For each critical point, determine, by the second-derivative test, whether it corresponds to a relative maximum, to a relative minimum, or to neither, or whether the test gives no information.

7. $f(x, y) = x^2 + 3y^2 + 4x - 9y + 3$

8. $f(x, y) = -2x^2 + 8x - 3y^2 + 24y + 7$

9. $f(x, y) = y - y^2 - 3x - 6x^2$

Above are from Haeussler Ch 17.6

Q7 Critical points:

$0 = f_x = 2x + 4$ hence $x = -2$

$0 = f_y = 6y - 9$ hence $y = 3/2$

$D = f_{xx}f_{yy} - (f_{xy})^2 = 12 > 0$ and $f_{xx} > 0$ so it's a minimum

Q

profit.

24. Profit Repeat Problem 23 if the constant costs of production of A and B are a and b (cents per lb), respectively.

25. Price Discrimination Suppose a monopolist is practicing price discrimination in the sale of a product by charging different prices in two separate markets. In market A the demand function is

$$p_A = 100 - q_A$$

and in B it is

$$p_B = 84 - q_B$$

where q_A and q_B are the quantities sold per week in A and B, and p_A and p_B are the respective prices per unit. If the monopolist's cost function is

$$c = 600 + 4(q_A + q_B)$$

how much should be sold in each market to maximize profit? What selling prices give this maximum profit? Find the maximum profit.

26. Profit A monopolist sells two competitive products, A and B, for which the demand functions are

$$q_A = 16 - p_A + p_B \quad \text{and} \quad q_B = 24 + 2p_A - 4p_B$$

If the constant average cost of producing a unit of A is 2 and a unit of B is 4, how many units of A and B should be sold to maximize

Solution

$c=600+4(100-p_A) + 4(84-p_B)$

G=proft

$=p_Aq_A + p_Bq_B - C= p_Aq_A + p_Bq_B+1336-4\ p_A-4p_B$

$= p_A(100-p_A)+ p_B(84-p_B)+1336-4\ p_A-4p_B$

$= 96\, p_A - p_A^{\,2} + 80\, p_B - p_B^{\,2} + 1336$

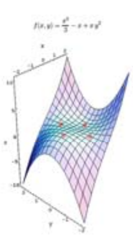
Find critical points:

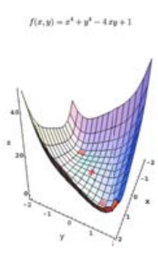
$$0 = \frac{\partial G}{\partial p_A} = 96 - 2p_A$$

$$0 = \frac{\partial G}{\partial p_B} = 80 - 2p_B$$

Critical point at prices (48, 40)

D>0 and $\frac{\partial^2 G}{\partial p_A^2} = -2$ so D is a maximum profit point





Prof Eric Moore on derivatives with >2 variables

Second Derivative Test Suppose $z = f(x_1, x_2, \cdots, x_n)$ has continuous second partial derivatives at all points $(x_1, x_2, \cdots, x_n)$ near a critical point $\boldsymbol{a} = (a_1, a_2, \cdots, a_n)$. Let

$$A = Hf(\boldsymbol{a}) = \begin{bmatrix} f_{x_1x_1}(\boldsymbol{a}) & f_{x_1x_2}(\boldsymbol{a}) & \cdots & f_{x_1x_n}(\boldsymbol{a}) \\ f_{x_1x_2}(\boldsymbol{a}) & f_{x_2x_2}(\boldsymbol{a}) & \cdots & f_{x_2x_n}(\boldsymbol{a}) \\ & \cdots & & \\ f_{x_1x_n}(\boldsymbol{a}) & f_{x_2x_n}(\boldsymbol{a}) & \cdots & f_{x_nx_n}(\boldsymbol{a}) \end{bmatrix}$$

Case 1: $\det A \neq 0$.

1. If $\det A_1 > 0$, $\det A_2 > 0$, $\det A_3 > 0$, $\cdots$ (i.e., if $\det A_k > 0$ for $1 \leq k \leq n$) then $f(a_1, a_2, \cdots, a_n)$ is a relative (local) minimum.

2. If $\det A_1 < 0$, $\det A_2 > 0$, $\det A_3 < 0$, $\cdots$ (i.e., if $\det A_k$ has sign $(-1)^k$ for $1 \leq k \leq n$) then $f(a_1, a_2, \cdots, a_n)$ is a relative (local) maximum.

3. For any other sequence, $f(a_1, a_2, \cdots, a_n)$ is neither a minimum nor a maximum.

Case 2: $\det A = 0$. This is the degenerate case. We can draw no conclusions — further analysis is required.

(Recall that A_k, $k = 1, 2, \cdots, n$ is the square matrix consisting of the first k rows and columns from A.)

Example with n=3

$f(x,y,z) = x^2y + yz + x^2 - 4z$

Critical points:

$0=f_x = 2xy + 2x$

$0=f_y= x^2 + z$

$0=f_z = y-4$

Hence $y=4$, $x=0$, $z=0$

Matrix of second derivatives, A:

$$A= \begin{bmatrix} 2y+2 & 2x & 0 \\ 2x & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$Det(A)=-2y-2 =- 10 <0$ *at* $(0,4,0)$

$A_1 = [2y+2]$

$Det(A_1) = 10 >0$

$$A_2 = \begin{bmatrix} 2y+2 & 2x \\ 2x & 0 \end{bmatrix}$$

$Det(A_2) = -4x^2 = 0$ *at (0,4,0)*

So (0,4,0) is neither a maximum nor a minimum.

Haeussler 17.7: Lagrange Multipliers

- **Lagrange multipliers** allow us to obtain critical points.
- The number λ_0 is called a **Lagrange multiplier.**

Example 1 – Method of Lagrange Multipliers

Find the critical points for

$z = f(x,y) = 3x - y + 6,$

subject to the constraint

$x^2 + y^2 = 4.$

Solution:

Constraint $g(x,y)=x^2 + y^2 -4=0$

Construct the function:

$F(x,y,\lambda)=f(x,y) - \lambda g(x,y)$

$= 3x - y +6 - \lambda(x^2 + y^2 -4)$

Setting $F_x = F_y = F_\lambda = 0$,gives

$0 = 3 - 2x\lambda$

$0 = -1 - 2y\lambda$

$0 = -x^2 - y^2 + 4$

Solve:

$x = 3/2\lambda$

$y = -1/2\lambda$

$\lambda = \pm 0.25\sqrt{10}$

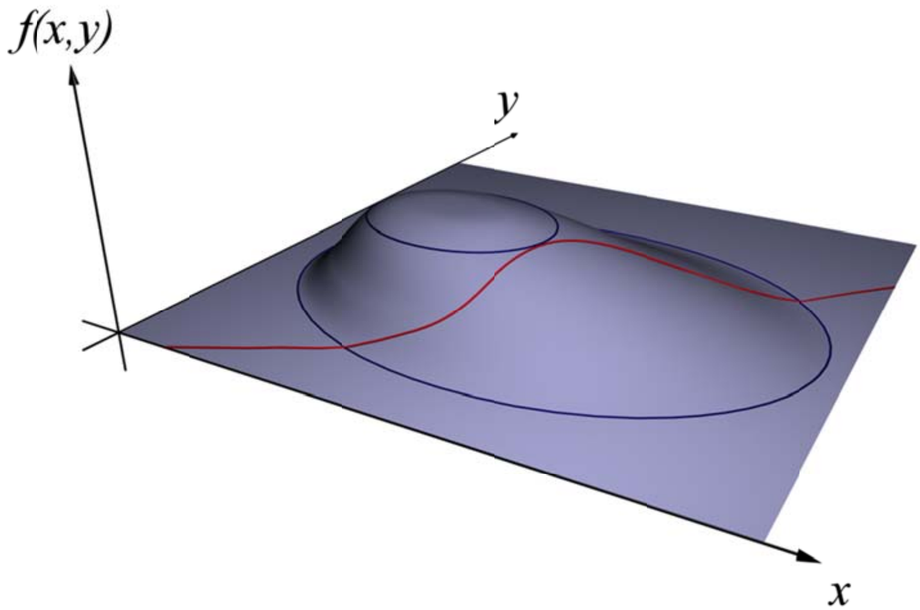
Critical points are

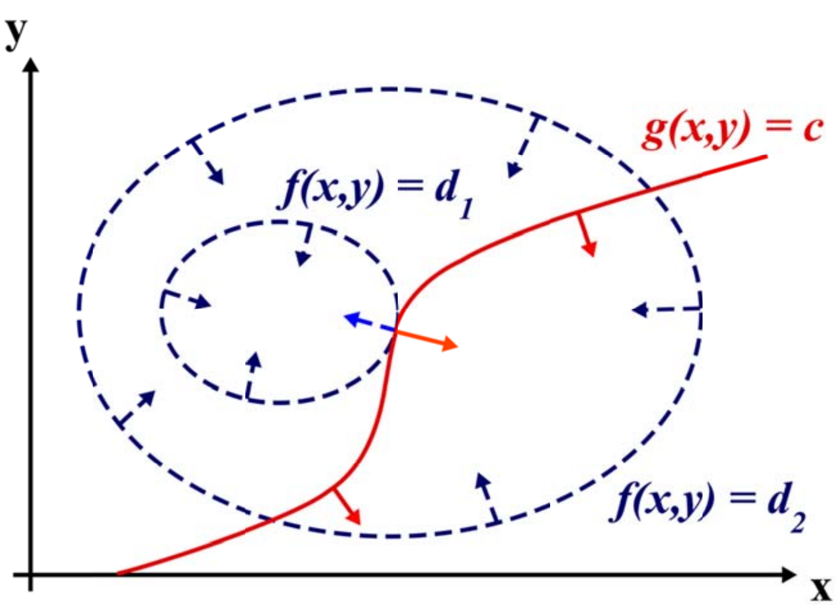
$a_1 = (6/\sqrt{10}, -2/\sqrt{10}, \sqrt{10}/4)$

$a_2 = (-6/\sqrt{10}, 2/\sqrt{10}, -\sqrt{10}/4)$

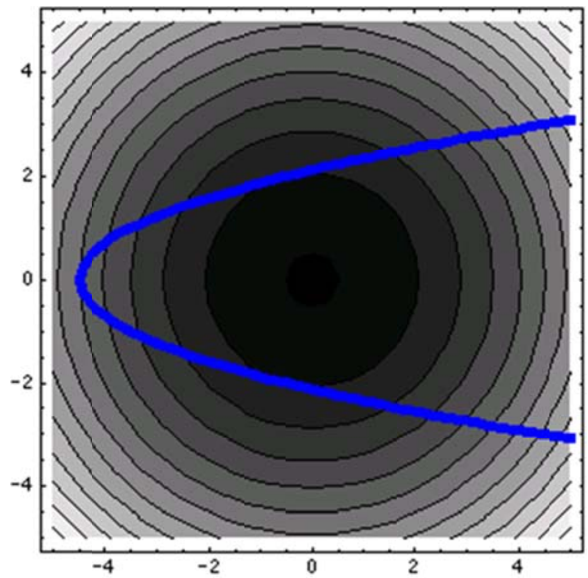
How did Lagrange dream this up?

Maybe think of being constrained to walk only on the footpath over the side of a hill. The crucial insight is that at the highest point on the footpath, the footpath is tangent to the level curves (height contours) of the hill.

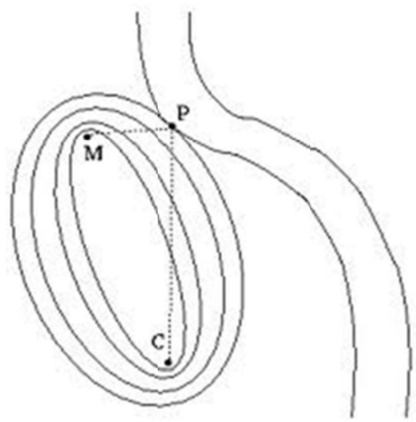




Lagrange's Dreams

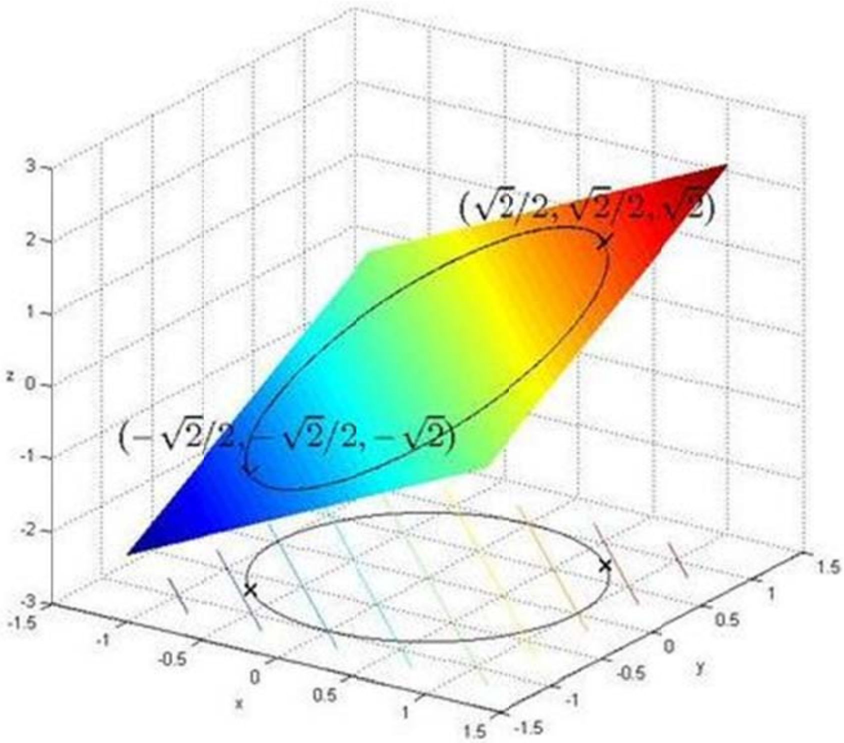


Above could be closest approach to the centre.



Above is a milkmaid getting from M (moo?) to C (cow?) via the riverbank P, where she rinses her pail. Ellipses are level curves of equal walking time-remember drawing ellipses with a loop of string?

Lagrange’s Dreams



Above is maximizing and minimizing height on the coloured plane, constrained by being on a path whose plan is a circle. The coloured lines on the ‘floor’ are level curves (contours, actually lines here since it’s a plane) of equal height. At the extrema, the circle is tangent to the coloured level curves. You might be able to see that making the circle a little bigger, hence increasing c and hence g(x,y) in

$x^2 + y^2 = c = g(x,y)$

will allow you to move at right angles to the tangent to get to the new optimum point.

Lagrange had more coffee and then…

Think of change in your height as you climb up the hill from the maximum point. Walking parallel to the path we increase neither g nor f. If walking at right angles to the path, you are increasing your g and your f as fast as possible, but f is increasing λ times faster than g. . If at an angle, both f and g are increasing at a lower rate, but the ratio λ is the same. Think of the plane tangent to the hill at the maximum point. Could choose the angle to take you parallel to the x axis,

$$\frac{\frac{\partial f}{\partial y}}{\frac{\partial g}{\partial y}} = \lambda$$

and then one taking you parallel to the y axis.

$$\frac{\frac{\partial f}{\partial x}}{\frac{\partial g}{\partial x}} = \lambda, \text{ same } \lambda.$$

Actually the 'same ratio any direction' behavior applies anywhere, not just at the extremum.

Lagrange multiplier

He gave the ratio of the rates a name, probably ‘that quotient I made up on Tuesday’, but in French, and then got a friend to start calling it ‘Lagrange multiple’ λ

$$\frac{\frac{\partial f}{\partial x}}{\frac{\partial g}{\partial x}} = \frac{\frac{\partial f}{\partial y}}{\frac{\partial g}{\partial y}} = \lambda$$

So that gave him the idea to set up a function $F(x,y,\lambda)=f(x,y) - \lambda g(x,y)$

so that if we take its derivatives we get back to his quotient:

$0=\frac{\partial F}{\partial x}$ $gives\ \frac{\partial f}{\partial x} = \lambda\frac{\partial g}{\partial x}$

$0=\frac{\partial F}{\partial y}$ $gives\ \frac{\partial f}{\partial y} = \lambda\frac{\partial g}{\partial y}$

$0=\frac{\partial F}{\partial \lambda}$ $gives\ the\ necessary\ third\ equation.$

We solve for critical point x,y,λ but rarely use λ. But λ has to be maximized so that we are indeed at the extremum point on the path.

Example from MATA33 Final W09

4. A rectangular box has four sides, a top, and a base. The material costs are $\$6/m^2$ for the sides, $\$4/m^2$ for the top, and $\$14/m^2$ for the base. A budget of exactly $\$6K^2$ is used for the total material cost ($K > 0$ is a constant) for the box. Use the Lagrange multiplier technique to find the length, width, and height of the box of maximum volume subject to the constraint of the material cost above. (You may assume that the critical point obtained by the Lagrange multiplier technique actually does result in a maximum volume.) [12 points]

Maximise $V=xyz$

subject to $2z(6x+6y) + 4xy + 14xy = 6K^2$

$F = xyz - \lambda(12xz + 12yz + 18xy - 6K^2)$

Can see symmetry here: $x=y$

$F = x^2z - \lambda(24xz + 18x^2 - 6K^2)$

$0 = F_x = 2xz - \lambda(24z + 36x)$

$0 = F_z = x^2 - \lambda(24x)$

$0 = -F_\lambda = 24xz + 18x^2 - 6K^2$

Hence

$0 = xz - \lambda(12z + 18x)$ (Eqn 1)

$0 = x^2 - \lambda(24x)$ (hence $x=24\lambda$ since $x=0$ is a minimum)

$0 = 4xz+3x^2-K^2$

Substituting $\lambda=x/24$

$0=xz - zx/2 - 3x^2/4$

$0=4xz+3x^2-K^2$

and rewriting

$0=x\,(2z-3x)$, hence $z=3x/2$

$0=9x^2-K^2$

$x=K/3$, $y=K/3$, $Z = K/2$

Example from MATA33 Final W08

Use the method of Lagrange multipliers to find the maximum value of $f(x; y) = xy + 2x$ subject to the constraint $2x+y = 30$ (You may assume that the critical point obtained does correspond to a maximum). [7 points]

Solution

$F(x,y)=xy+2x - \lambda(2x+y-30)$

$$0 = \frac{\partial F}{\partial x} = y + 2 - 2\lambda$$

$$0 = \frac{\partial F}{\partial y} = x - \lambda$$

$$0 = -\frac{\partial F}{\partial \lambda} = 2x + y - 30$$

Eqn(1) and (2) give: $0=y+2-2x$

Solve: $y=14$, $x=8$, *maximum* f *is* $8*14+2*8=128$

Example 3 – Minimizing Costs

Suppose a firm has an order for 200 units of its product and wishes to distribute its manufacture between two of its plants, plant 1 and plant 2. Let q_1 and q_2 denote the outputs of plants 1 and 2, respectively, and suppose the total-cost function is given by

$$c = f(q_1, q_2) = 2q_1^2 + q_1 q_2 + q_2^2 + 200.$$

How should the output be distributed in order to minimize costs?

Solution: We minimize $c = f(q_1, q_2)$, given the constraint $q_1 + q_2 = 200$.

$$F(q_1, q_2, \lambda) = 2q_1^2 + q_1 q_2 + q_2^2 + 200 - \lambda(q_1 + q_2 - 200)$$

$$\begin{cases} \dfrac{\partial F}{\partial q_1} = 4q_1 + q_2 - \lambda = 0 \\ \dfrac{\partial F}{\partial q_2} = q_1 + 2q_2 - \lambda = 0 \\ \dfrac{\partial F}{\partial \lambda} = -q_1 - q_2 + 200 = 0 \end{cases}$$

Solve to get $q_1 = 50$, $q_2 = 150$

Example 5 – Method of Lagrange Multipliers with Two Constraints

Find critical points for f(x, y, z) = xy+ yz, subject to the constraints $x^2 + y^2 = 8$ and yz = 8.

Solution:

$$F(x, y, z, \lambda_1, \lambda_2) = xy + yz - \lambda_1 (x^2 + y^2 - 8) - \lambda_2 (yz - 8)$$

$$\begin{cases} F_x = y - 2x\lambda_1 = 0 \\ F_y = x + z - 2y\lambda_1 - z\lambda_2 = 0 \\ F_z = y - y\lambda_2 = 0 \\ F_{\lambda_1} = -x^2 - y^2 + 8 = 0 \\ F_{\lambda_2} = -yz + 8 = 0 \end{cases}$$

$$\begin{cases} \dfrac{y}{2x} = \lambda_1 \\ x + z - 2y\lambda_1 - z\lambda_2 = 0 \\ \lambda_2 = 1 \\ x^2 + y^2 = 8 \\ z = \dfrac{8}{y} \end{cases}$$

We obtain 4 critical points:

(2, 2, 4)   (2,−2,−4)   (−2, 2, 4)   (−2,−2,−4)

Haussler Ch 17.9 Multiple Integrals

Definite integrals of functions of two variables are called (definite) **double integrals**, which involve integration over a *region* in the plane.

Example 1 – Evaluating a Double Integral

Find $\displaystyle\int_{-1}^{1}\int_{0}^{1-x} (2x+1)\ dy\ dx.$

Solution:

$$\int_{-1}^{1}\int_{0}^{1-x} (2x+1)\ dy\ dx$$

$$= \int_{-1}^{1} \left[2xy + y\right]_0^{1-x} dx$$

$$= \int_{-1}^{1} \left[2x(1-x) + 1 - x\right]_0^{1-x} dx$$

$$= \left[-\frac{2x^3}{3} + \frac{x^2}{2} + x\right]_{-1}^{1} = \frac{2}{3}$$

Example 3 – Evaluating a Triple Integral

Find $\int_0^1 \int_0^x \int_0^{x-y} x \, dz \, dy \, dx.$

Solution:

$$\int_0^1 \int_0^x \int_0^{x-y} x \, dz \, dy \, dx = \int_0^1 \int_0^x \left[xz \right]_0^{x-y} \, dy \, dx$$

$$= \int_0^1 \int_0^x \left[x(x-y) \right] \, dy \, dx$$

$$= \int_0^1 \left[x^2 y - \frac{xy^2}{2} \right]_0^x dx$$

$$= \int_0^1 \left[x^3 - \frac{x^3}{2} \right]_0^x dx$$

$$= \left[\frac{x^4}{8} \right]_0^1 = \frac{1}{8}$$

Question 9, MATA33 Final W08

9. (a) Evaluate $I = \int_1^e \int_0^x ln(x)\ dydx$

Solution:

$$I = \int_1^e \int_0^x \ln(x)\, dydx$$

$$=\int_1^e [y \ln(x)]_{y=0}^{x} dx$$

$$= \int_1^e [x \ln(x)]\ \ dx$$

$$= \int_1^e [x \ln(x)]\ \ dx \qquad \textit{(use } \int udv = [uv] - \int vdu\textit{)}$$

$$= [0.5x^2\, ln(x)]_1^e - \int_1^e [0.5x^2/\text{x}]\ \ dx$$

(with $v=x^2/2$, $u= ln(x)$)

$$=0.5e^2 - 0.25(e^2 - 1)= 0.25(e^2 + 1)$$

Question 9, MATA33 Final W08

(b) Evaluate $J = \int\int_R y^2 e^{xy}\, dA$ where R is the triangular region with vertices $(0,0)$, $(1,1)$, and $(0,1)$. [7 points]

$$J=\int_{y=0}^{1} \int_{x=y}^{1} y^2\, e^{xy} dx dy$$

Type equation here.

$$= \int_{y=0}^{1} [y\, e^{xy}]\,_{x=y}^{1} dy$$

$$= \int_{y=0}^{1} [y\,(e^{\,y} - \ e^{y^2})]\ \ dy$$

(substitute $z=y^2$, $dz/dy=2y$)

$$= \int_{y=0}^{1} [y\, e^{\,y}\ \ ]\ \ dy\ \ - \int_{y=0}^{1} [y\ \ e^{y^2}]\ \ dy$$

$$= [y\, e^{\,y}]_{y=0}^{1} - \int_{y=0}^{1} [e^{\,y}]\, dy - 0.5\left[y^{2}\, e^{y^{2}}\right]_{y=0}^{1}$$

$$+ \quad 0.5 \int_{y=0}^{1} [\, e^{y^{2}}]\, d(y^{2})$$

$= e - e + 1 - 0.5e + 0.5\ [e - 1]$

$$= 0.5$$