ACT466H1S - TEST 3 - APRIL 12, 2007

Write name and student number on each page. Write your solution for each question in the space provided.

1. S has a compound distribution.

The frequency distribution N has probability function

n :	0	1	2	3
$P(N = n)$:	.4	.3	.2	.1

The severity (single claim) distribution Y has probability function

y :	1	2	3
$P(Y = y)$:	.5	.3	.2

5 (a) S is to be simulated in two steps. In the first step, N is simulated using the usual inverse transform method using a uniform (0,1) random number $u = .75$. In the second step, whatever the simulated number of claims is from step 1, that many claim amounts (Y) are simulated using as many of the following uniform (0,1) random numbers as necessary: $.35$, $.82$, $.41$.
Find the simulated value of S.

5 (b) S is to be simulated using the inverse transform method applied directly to S.
Using a uniform (0,1) number of .65, find the simulated value of S.

2.. The geometric distribution with parameter p ($0 < p < 1$) has probability function
$P(X = k) = p(1-p)^k$, $k = 0, 1, 2, ...$
An alternative parametrization of this distribution has $\theta = \frac{1-p}{p}$, and
$P(X = k) = \frac{\theta^k}{(1+\theta)^{k+1}}$, $k = 0, 1, 2, ...$
Show that each of the following methods is a valid method of simulating this geometric random variable X by showing that the probability function of the simulated random variable Y is the same as that of X.

3 (a) Method 1: This method requires successive independent uniform (0,1) numbers $u_1, u_2, ...$
For u_1 , set $Y = 0$ if $u_1 < p$.
If $u_1 \geq p$, and if $u_2 < p$, then set $Y = 1$.
In general, if $u_i \geq p$ for $i = 1, 2, ..., k$ and if $u_{k+1} < p$, then set $Y = k$
(where k is an integer ≥ 1).

2 (b) Method 2: Given uniform (0,1) random number u, find $\frac{ln(1-u)}{ln(1-p)}$.
If k is an integer ≥ 0 such that $k \leq \frac{ln(1-u)}{ln(1-p)} < k+1$, then set $Y = k$.

2 (c) Method 3: This is a two step method. In the first step, a uniform (0,1) random number u is used to find $\lambda = -\theta \, ln(1-u)$. In the second step, a second uniform (0,1) random number v (independent of u) is used with the usual inverse transform method to simulate a Poisson random variable with mean λ. Y is set equal to the value of the simulated Poisson random variable. The second simulation is conditional on the value of the first simulation.

3. The random variable X has pdf $\frac{1}{2\sqrt{x}}$ for $0 < x < 1$.

2 (a) Given uniform (0,1) random number u, apply the inverse transform method to simulate X. Express the simulated value of X in terms of u.

2 (b) If u is uniformly distributed on (0,1), show that $v = 1 - u$ also has a uniform distribution on (0,1).

(c) For a given uniform (0,1) random number u, apply the inverse transform method to simulate x_1 from u and to simulate x_2 from $1 - u$.

2 (i) Find the covariance between x_1 and x_2. Are x_1 and x_2 independent?

2 (ii) Find the variance of $x_1 + x_2$. How does it compare to the variance of the sum of two independent values of X?

4. The following random sample of size 5 is taken from the distribution of X:

1 , 3 , 4 , 7 , 10

Bootstrap approximation of the mean square error of estimators is to be based on the following 7 resamplings of size 5 from the empirical distribution:

Resample 1 : 1 , 1 , 4 , 7 , 7
Resample 2 : 3 , 4 , 4 , 7 , 10
Resample 3 : 1 , 4 , 4 , 10 , 10
Resample 4 : 3 , 3 , 3 , 4 , 10
Resample 5 : 4 , 4 , 7 , 7 , 10
Resample 6 : 1 , 7 , 7 , 10 , 10
Resample 7 : 4 , 7 , 7 , 7 , 10

The median of X is estimated by the average of the smallest and largest sample values, and the 3rd moment of X is estimated by the estimator $\frac{1}{5}\sum_{i=1}^{5} X_i^3$.

5+5 Find the bootstrap approximation to each of those estimators using the 7 resamplings.

5. X has pdf $f(x) = xe^{-x}$ and cdf $F(x) = 1 - (x+1)e^{-x}$.
You are given that the 90th percentile of X is 3.8897 .
Find the 90th percentile conditional tail expectation, $CTE_{.9}$.

6. X has a geometric distribution with $p = \frac{1}{2}$, and probability function $P(X=k)=(1-p)^k p$, $k=0,1,2,...$. The mean of X is 1.

5 (a) Using the definition of percentile for a discrete random variable in the Risk Measures study note, find the 90th percentile of X.

5 (b) Find the 90th percentile conditional tail expectation for X, $CTE_{.9}$.

ACT466H1S - TEST 2 - APRIL 12, 2007 - SOLUTIONS

1.(a) The cdf of N is

$n:$	0	1	2	3
$P(N \leq n):$	.4	.7	.9	1

The uniform value of .75 results in a simulated value of N of 2, since $.7 \leq .75 < .9$.

We must simulate 2 values of Y. The cdf of Y is

$y:$	1	2	3
$P(Y \leq y):$	.5	.8	1

The first uniform number .35 results in a simulated value of $Y_1 = 1$,
and the second uniform number .82 results in $Y_2 = 3$.
The simulated value of S is $1 + 3 = 4$.

(b) The probability function of S is found by considering the various combinations of N and Y.

$P(S = 0) = P(N = 0) = .4 \rightarrow F_S(0) = .4$.

$P(S = 1) = P(N = 1) \cdot P(Y = 1) = (.3)(.5) = .15 \rightarrow F_S(1) = .55$.

$P(S = 2) = P(N = 1) \cdot P(Y = 2) + P(N = 2) \cdot [P(Y = 1)]^2$
$= (.3)(.3) + (.2)(.5)^2 = .14 \rightarrow F_S(2) = .69$.

For the uniform (0,1) number .65, we have $F_S(1) = .55 \leq .65 < .69 = F_S(2)$.
The simulated value of S is 2.

2.(a) For any u, $P(u < p) = p$ and $P(u > p) = 1 - p$ because u is uniformly distributed/
The probability of k successive u-values $> p$ followed by a u-value less than p is
$(1-p)^k p$, which is the probability that $X = k$.

(b) $P(Y = k) = P(k \leq \frac{ln(1-u)}{ln(1-p)} < k + 1)$
$= P[k\, ln(1-p) \geq ln(1-u) > (k+1)\, ln(1-p)]$
$= P[(1-p)^k \geq 1 - u > (1-p)^{k+1}]$
Since u is uniformly distributed on (0,1), so is $1 - u$.
$P[(1-p)^k \geq 1 - u > (1-p)^{k+1}]$
$= (1-p)^k - (1-p)^{k+1} = (1-p)^k \cdot p$.

Alternatively,
$P[(1-p)^k \geq 1 - u > (1-p)^{k+1}]$
$= P[1 - (1-p)^k \leq u < 1 - (1-p)^{k+1}]$
$= 1 - (1-p)^{k+1} - [1 - (1-p)^k] = (1-p)^k - (1-p)^{k+1} = (1-p)^k \cdot p$.

(c) The first simulation results in a value of λ from the exponential distribution with mean θ,
is the pdf of λ is $\pi(\lambda) = \frac{1}{\theta} e^{-\lambda/\theta}$.
The second simulation has probability function $P(Y = k|\lambda) = \frac{e^{-\lambda}\lambda^k}{k!}$.
The unconditional probability $P(Y = k)$ is
$P(Y = k) = \int_0^\infty P(Y = k|\lambda) \cdot \pi(\lambda)\, d\lambda = \int_0^\infty \frac{e^{-\lambda}\lambda^k}{k!} \cdot \frac{1}{\theta} e^{-\lambda/\theta}\, d\lambda$
$= \frac{1}{\theta k!} \int_0^\infty \lambda^k e^{-\lambda(\frac{\theta+1}{\theta})}\, d\lambda = \frac{1}{\theta k!} \cdot \frac{k!}{(\frac{\theta+1}{\theta})^{k+1}} = \frac{\theta^k}{(1+\theta)^{k+1}}$.

3.(a) $F_X(t) = \int_0^t \frac{1}{2\sqrt{x}}\,dx = \sqrt{t}$, $0 < t < 1$.

According to the inverse transform method, we solve for x from
$u = F_X(x) = \sqrt{x}$, so that $x = u^2$.

(b) We can apply the usual Jacobian method to find the pdf of v:
$v = 1 - u$, and $f_U(u) = 1$ for $0 < u < 1$
→ $f_V(v) = f_U(1 - v) \cdot |\frac{d}{dv}(1 - v)| = 1$ for $0 < 1 < v$.

Alternatively, $F_V(t) = P[V < t] = P[1 - U < t] = P[U > 1 - t] = t$.
so $f_V(t) = 1$.

(c)(i) $x_1 = u^2$, $x_2 = (1 - u)^2$.
$Cov(x_1, x_2) = Cov(u^2, (1 - u)^2) = E[u^2(1 - u)^2] - E[u^2]E[(1 - u)^2]$.
$E[u^2] = \int_0^1 u^2\,du = \frac{1}{3} = E[(1 - u)^2] = E[X]$.
$E[u^2(1 - u)^2] = E[u^2 - 2u^3 + u^4] = \frac{1}{3} - 2(\frac{1}{4}) + \frac{1}{5} = \frac{1}{30}$.
$Cov(x_1, x_2) = \frac{1}{30} - (\frac{1}{3})(\frac{1}{3}) = -\frac{7}{90}$. x_1 and x_2 are not independent

(ii) $Var[x_1 + x_2] = Var(x_1) + Var(x_2) + 2Cov(x_1, x_2)$.
$Var(x_1) = E[X^2] - (E[X])^2$.
$E[X^2] = \int_0^1 x^2 \cdot \frac{1}{2\sqrt{x}}\,dx = \frac{1}{5}$, and $E[X] = \frac{1}{3}$.
$Var[X] = \frac{1}{5} - (\frac{1}{3})^2 = \frac{4}{45}$.
$Var[x_1 + x_2] = \frac{4}{45} + \frac{4}{45} + 2(-\frac{7}{90}) = \frac{1}{45}$.
The variance of the sum of two independent X's is $\frac{4}{45} + \frac{4}{45} = \frac{8}{45}$,
larger than the variance of $x_1 + x_2$ found from u.

4. Median:

The median of the empirical distribution is $\theta = 4$.

Resample	$\widehat{\theta}_1$	$(\widehat{\theta}_1 - 4)^2$
1 , 1 , 4 , 7 , 7	4	$(4-4)^2 = 0$
3 , 4 , 4 , 7 , 10	$\frac{13}{2}$	$(\frac{13}{2} - 4)^2 = \frac{25}{4}$
1 , 4 , 4 , 10 , 10	$\frac{11}{2}$	$(\frac{11}{2} - 4)^2 = \frac{9}{4}$
3 , 3 , 3 , 4 , 10	$\frac{13}{2}$	$(\frac{13}{2} - 4)^2 = \frac{25}{4}$
4 , 4 , 7 , 7 , 10	7	$(7-4)^2 = 9$
1 , 7 , 7 , 10 , 10	$\frac{11}{2}$	$(\frac{11}{2} - 4)^2 = \frac{9}{4}$
4 , 7 , 7 , 7 , 10	7	$(7-4)^2 = 9$

The bootstrap estimate of MSE$(\widehat{\theta})$ is $\frac{0+\frac{25}{4}+\frac{9}{4}+9+\frac{9}{4}+9}{7} = 5.0$.

3rd Central Moment:

The 3rd moment of the empirical distribution is $\frac{1}{5}[1^3 + 3^3 + 4^3 + 7^3 + 10^3] = 287$.

Resample	$\widehat{\theta}_1$	$(\widehat{\theta}_1 - 4)^2$
1 , 1 , 4 , 7 , 7	150.4	$(150.4 - 287)^2 = 18,659.56$
3 , 4 , 4 , 7 , 10	299.6	$(299.6 - 287)^2 = 158.76$
1 , 4 , 4 , 10 , 10	425.8	$(425.8 - 287)^2 = 19,265.44$
3 , 3 , 3 , 4 , 10	229	$(229 - 287)^2 = 3364$
4 , 4 , 7 , 7 , 10	362.8	$(362.8 - 287)^2 = 5745.64$
1 , 7 , 7 , 10 , 10	537.4	$(527.4 - 287)^2 = 62,700.16$
4 , 7 , 7 , 7 , 10	418.6	$(418.6 - 287)^2 = 17,318.56$

The bootstrap estimate of MSE$(\widehat{\theta})$ is $18,173.16$.

5. Since X is continuous, $CTE_{.9} = \frac{E[X]-E[X\wedge Q_{.9}]}{1-.9} + Q_{.9}$,

where $Q_{.9} = 3.8897$ is the 90th percentile of X.

$E[X] - E[X \wedge Q_{.9}] = E[X] - E[X \wedge 3.8897] = \int_{3.8897}^{\infty}[1 - F(x)]\,dx$

$= \int_{3.8897}^{\infty}(x+1)e^{-x}\,dx = -(x+2)e^{-x}\Big|_{x=3.8897}^{x=\infty} = (3.8897+2)e^{-3.8897} = .12045$.

Then, $CTE_{.9} = \frac{.12045}{.1} + 3.8897 = 5.09$.

6. (a) $Q_{.9} = min\{Q : P(X \leq Q_{.9}) \geq .9\}$.

$F_X(k) = p + (1-p)p + \cdots + (1-p)^k p = 1 - (1-p)^{k+1}$.

To find the 90th percentile, we want $f(k) = 1 - \frac{1}{2^{k+1}} \geq .9$,

so that $\frac{1}{2^{k+1}} \leq .1$. We see that $\frac{1}{2^{3+1}} = .0625 < .1 < .125 = \frac{1}{2^{2+1}}$.

The 90th percentile of X is $Q_{.9} = 3$, since $P(X \leq 3) = .5 + .25 + .125 + .0625 = .9375 \geq .9$ and $P(X \leq 2) = .875 < .9$.

(b) $CTE_{.9} = \frac{(\beta' - .9)Q_{.9} + (1-\beta')E[X|X>Q_{.9}]}{1-.9}$.

From part (a), we have $Q_{.9} = 3$.

$\beta' = max\{\beta : Q_{.9} = Q_\beta = 3\}$. From part (a), we see that $Q_{.9375} = 3$ since $P(X \leq 3) = .9375$, and $Q_\beta = 4$ for any $\beta > .9375$.

Therefore, $\beta' = .9375$.

$CTE_{.9} = \frac{(.9375-.9)(3)+(1-.9375)E[X|X>3]}{.1}$.

$E[X|X>3] = \frac{4P(X=4)+5P(X=5)+\cdots}{P(X>3)}$.

But

$4P(X=4) + 5P(X=5) + \cdots = E[X] - P(X=1) - 2P(X=2) - 3P(X=3)$

$= 1 - .25 - 2(.125) - 3(.0625) = .3125$, so that

$E[X|X>3] = \frac{.3125}{.0625} = 5$.

Then, $CTE_{.9} = \frac{(.9375-.9)(3)+(1-.9375)(5)}{.1} = 4.25$.