MARKOV CHAINS - ADDITIONAL PROBLEMS

1. A two-state homogeneous Markov chain is being used to model the transitions between days with rain (R) and without rain (N). You are given $Q^{(R,R)} = .5$, $Q^{(N,N)} = .75$.

(a) If it is raining today, find the probability it is raining two days from today.

(b) If it is raining today, find the expected number of non-rainy days until the next rainy day.

(c) Find the eigenvalues and the eigenvectors of $\boldsymbol{Q}$ (they should be 1 and $\frac{1}{4}$).

Use the representation to show that an expression for ${}_n\boldsymbol{Q}$ is $\begin{bmatrix} \frac{1}{3} + \frac{2}{3\times\cdot 4^n} & \frac{2}{3} - \frac{2}{3\times 4^n} \\ \frac{1}{3} - \frac{1}{3\times 4^n} & \frac{2}{3} + \frac{1}{3\times 4^n} \end{bmatrix}$

2. A homogeneous Markov chain $\{X_n : n \geq 0\}$ has states 0,1,2. The one-step transition probability matrix is $\boldsymbol{Q} = \begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

(a) Find the transition probabilities ${}_2Q^{(2,0)}$ and ${}_3Q^{(2,0)}$.

(b) Suppose that the distribution of X_0 is $P[X_0 = 0] = P[X_0 = 1] = \frac{1}{2}$ and $P[X_0 = 2] = 0$. Find the unconditional distribution of X_1 .

(c) Show that all states of this Markov chain communicate with one another.

(d) Suppose the process is now in state 2. Find the expected number of transitions until the process is in state 2 again.

(e) Find the probability $P((X_3 = 2) \cap (X_2 \neq 0) \cap (X_1 \neq 0) | X_0 = 1)$

3. Residents in a nursing home are classified according to one of three states:
1 - living independently in the nursing home,
2 - living in the health care unit of the home, and
3 - no longer in the home.
Transitions between the states occur at the end of each year, and are modeled according to a Markov chain with the following one-step transition probability matrix: $\begin{bmatrix} .6 & .2 & .2 \\ .4 & .4 & .2 \\ 0 & 0 & 1 \end{bmatrix}$.

A new resident is living independently in the nursing home.
(a) Find the expected number of years that the resident will live in the home.
(b) Find the expected number of years the resident will spend living independently in the home before leaving the home.
(c) Find the expected number of years the resident will spend living in the health care unit before leaving the home.

4. A two-state Markov chain has the following transition probability matrix $\boldsymbol{P} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$.

The states are $\{1, 2\}$.
(a) If the chain is currently in state 1, find the expected number of transitions until it returns to state 1.
(b) If the chain is currently in state 1, find the expected number of visits to state 2 before the next return to state 1.
(c) Find the eigenvalues and eigenvectors of $\boldsymbol{Q}$ (the eigenvalues should be 1 and $\frac{1}{12}$).
Find the diagonal representation of the one step transition probability matrix in the form $\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} = \boldsymbol{U}\,\boldsymbol{D}\boldsymbol{U}^{-1}$ (find $\boldsymbol{U}$, $\boldsymbol{U}^{-1}$ and $\boldsymbol{D}$).

Use the representation to show that an expression for ${}_n\boldsymbol{Q}$ is $\begin{bmatrix} \frac{3}{11} + \frac{8}{11\cdot 12^n} & \frac{8}{11} - \frac{8}{11\cdot 12^n} \\ \frac{3}{11} - \frac{3}{11\cdot 12^n} & \frac{8}{11} + \frac{3}{11\cdot 12^n} \end{bmatrix}$.

The chain is in state 1 at time 0. Find the probability $P[X_{1000} = 1 \cap X_{1001} = 1 | X_0 = 1]$.

Bonus: Show by mathematical induction that ${}_n\boldsymbol{Q} = \begin{bmatrix} \frac{3}{11} + \frac{8}{11\cdot 12^n} & \frac{8}{11} - \frac{8}{11\cdot 12^n} \\ \frac{3}{11} - \frac{3}{11\cdot 12^n} & \frac{8}{11} + \frac{3}{11\cdot 12^n} \end{bmatrix}$

5. A 3-year college names some students to the dean's honour list at the end of each year. The student might or might not be named to the list at the end of the 2nd and the end of the 3rd year. Some students also drop out each year. The college models student behaviour according to a non-homogeneous Markov Chain. At the start of each year, each student is classified as being in one of three states:
1 - not on the honour list at the end of the previous year,
2 - on the honour list at the end of the previous year, and
3 - dropped out before the end of the previous year.
Every new student starts out in state 1.
It is assumed that once a student drops out, the student does not return.

For the 3 years that the student is in college, we have the following one-step transition matrices:

1st year $\boldsymbol{Q_0} = \begin{bmatrix} .5 & .25 & .25 \\ - & - & - \\ - & - & - \end{bmatrix}$, 2nd year $\boldsymbol{Q_1} = \begin{bmatrix} .6 & .2 & .2 \\ .4 & .5 & .1 \\ 0 & 0 & 1 \end{bmatrix}$,

3rd year $\boldsymbol{Q_2} = \begin{bmatrix} .8 & .1 & .1 \\ .25 & .75 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) The Dean's Honour List award pays $1000 to each student that is named to the list at the end of each year. Find the total expected amount of scholarship that will be awarded to a new student over his 3 year college career from the Dean's Honour List Award.

(b) The Most Improved Student award pays $1000 to each student that is named to the honour list at the end of a year if the student had not been on the list at the end of the previous year. This award is only made at the end of 2nd and 3rd years. For a new student in the college, find the expected amount of this award that will be received by the student over his 3 year college career.

(c) Find the probability that a new student will be named to the honour list at least once during his college career.

6. A markov chain $\{X_n : n \geq 0\}$ has states 0,1,2. The one-step transition probability matrix is

$\boldsymbol{Q} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

(a) Find the transition probabilities ${}_2Q^{(2,0)}$ and ${}_3Q^{(2,0)}$.
(b) Suppose that the distribution of X_0 is $P[X_0 = 0] = P[X_0 = 1] = P[X_0 = 2] = \frac{1}{3}$.
Find the unconditional distributions of X_1 and X_2 .
(c) Find the probability $P[X_3 = 1,\, X_2 \neq 2,\, X_1 \neq 2 \,|\, X_0 = 1]$

MARKOV CHAINS - SUPPLEMENTARY PROBLEM SET

7. A Gambler begins a gambling game with 2 dollars.
The game ends when he either goes broke (reaches 0) or reaches 3 dollars.
The gambler can win or lose one on each bet, but the win-lose probabilities depend on the amount of money he has.

Amount the gambler has	Prob. win	Prob. lose
1	$Q^{(1,2)} = 2/3$	$Q^{(1,0)} = 1/3$
2	$Q^{(2,3)} = 1/3$	$Q^{(2,1)} = 2/3$

(a) Formulate the one-step transition probability matrix for the Markov Chain representation of this process.
(b) Find the probability that the gambler gets to 3 before going broke.
(c) Find the expected number of bets until the gambler either gets to 3 or goes broke.
(d) Find the expected number times in the future that the gambler will have 2 dollars.

8. An auto insurance company has a rating system which rates drivers as Low, Medium, or High risk. A rating is assigned to the policyholder at the time an auto insurance policy is issued. The rating is updated at the end of each year. The transition probability matrix of risk rating in a policyholder's first year is

$$\begin{array}{cccc} & L & M & H \\ L & .6 & .3 & .1 \\ M & .3 & .4 & .3 \\ H & .1 & .2 & .7 \end{array} .$$

For all years after the first year, the transition probability matrix of risk rating is

$$\begin{array}{cccc} & L & M & H \\ L & .8 & .1 & .1 \\ M & .2 & .6 & .2 \\ H & 0 & .1 & .9 \end{array} .$$

(a) Suppose that a new policyholder is rated as low risk. Find the probability that this policyholder will never be rated as high risk by the end of (and including) the third policy year.

(b) Based on a policyholder's rating at the start of a year, the insurance company estimates annual claims for that year as follows (assumed to be paid at the end of each year)
Low risk: \$100 ; Medium Risk: \$200 ; High Risk: \$1000 .
For a new policyholder rated Low risk, find the total expected claims for the first three years.

(c) The insurer only issues a new policy to a low risk driver, but the insurer will renew the policy if that driver changes risk rating in subsequent years. The insurer charges a premium of \$125 at the time the policy is issued. The insurer will charge premiums in the following way at the start of the 2nd and 3rd year based on the risk rating:
2nd and 3rd year premiums: Low risk: \$125 ; Medium risk: \$250 ; High risk : $\$K$.
(i) Find the value of K based on the equivalence principle, for which the expected value of claims for the three year period is equal to the expected value of premiums for the 3 year period.
(ii) Using the value of K found in part (i), find the reserve on the policy at the end of the first year for a driver who is rated as Medium risk at the end of the first year.

(d) The insurer only issues a new policy to a low risk driver, but the insurer will renew the policy if that driver changes risk rating in subsequent years. The insurer charges a policy renewal fee if the policyholder's risk rating declines. The policy renewal fee is charged at the end of the year when the new rating is determined. The policy renewal fees are as follows:
change from Low to Medium: $50
change from Low to High: $200
change from Medium to High: $100
Find the expected policy renewal fees to be charged

MARKOV CHAINS - SOLUTIONS TO ADDITIONAL PROBLEMS

1. $Q = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$

(a) $_2Q^{(R,R)} = Q^{(R,R)} \cdot Q^{(R,R)} + Q^{(R,N)} \cdot Q^{(N,R)} = (\frac{1}{2})(\frac{1}{2}) + (\frac{1}{2})(\frac{1}{4}) = \frac{3}{8}$.

(b) Let c be the expected number of non-rainy days until the next rainy day if it is raining today, and let d be the expected number of future non-rainy days until the next rainy day if it is not raining today. Conditioning over the weather tomorrow, we have
$c = (0) \cdot Q^{(R,R)} + (1+d) \cdot Q^{(R,N)} = \frac{1}{2}d + \frac{1}{2}$.
We get an equation for d in the same way
$d = (0)Q^{(N,R)} + (1+d)Q^{(N,N)} = \frac{3}{4}d + \frac{3}{4}$. From this equation we get $d = 3$.
Then from the earlier equation we get $c = 2$.

(c) The eigenvalues are the solutions of $det\begin{bmatrix} \frac{1}{2} - v & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} - v \end{bmatrix} = 0$, so that

$(\frac{1}{2} - v)(\frac{3}{4} - v) - (\frac{1}{2})(\frac{1}{4}) = 0$. Solving for v results in $v = 1, \frac{1}{4}$, the eigenvalues.

The eigenvector for $v = 1$ is of the form $\begin{bmatrix} a \\ b \end{bmatrix}$, where $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$.

Then $\frac{1}{2}a + \frac{1}{2}b = a \rightarrow a = b \rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the basic eigenvector for $v = 1$.

The eiegenvector for $v = \frac{1}{4}$ is of the form $\begin{bmatrix} a \\ b \end{bmatrix}$, where $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \frac{1}{4}a \\ \frac{1}{4}b \end{bmatrix}$.

Then $\frac{1}{2}a + \frac{1}{2}b = \frac{1}{4}a \rightarrow a = -2b \rightarrow \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is the basic eigenvector for $v = \frac{1}{4}$.

Then $\boldsymbol{D} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$ and $\boldsymbol{U} = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$.

Finally, $\boldsymbol{U^{-1}} = \frac{1}{(1)(-1)-(2)(1)} \cdot \begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$.

$$\boldsymbol{Q} = \boldsymbol{U} \cdot \boldsymbol{D} \cdot \boldsymbol{U}^{-1}$$

$$\boldsymbol{Q^{(n)}} = \boldsymbol{U} \cdot \boldsymbol{D^n} \cdot \boldsymbol{U}^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4^n} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} + \frac{2}{3\times\cdot 4^n} & \frac{2}{3} - \frac{2}{3\times 4^n} \\ \frac{1}{3} - \frac{1}{3\times 4^n} & \frac{2}{3} + \frac{1}{3\times 4^n} \end{bmatrix}$$

2.(a) $\boldsymbol{Q^2} = \begin{bmatrix} \frac{2}{9} & \frac{7}{9} & 0 \\ \frac{2}{9} & \frac{1}{3} & \frac{4}{9} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix} \rightarrow {}_2Q^{(2,0)} = \frac{2}{3}$,

or alternatively,

${}_2Q^{(2,0)} = Q^{(2,0)} \cdot Q^{(0,0)} + Q^{(2,1)} \cdot Q^{(1,0)} + Q^{(2,2)} \cdot Q^{(2,0)} = 0 + (1)(\frac{2}{3}) + 0 = \frac{2}{3}$.

$\boldsymbol{Q^3} = \begin{bmatrix} \frac{14}{27} & \frac{1}{3} & \frac{4}{27} \\ \frac{2}{9} & \frac{17}{27} & \frac{4}{27} \\ \frac{2}{9} & \frac{1}{3} & \frac{4}{9} \end{bmatrix} \rightarrow {}_3Q^{(2,0)} = \frac{2}{9}$;

alternatively, this is the row 2 column 0 entry in $\boldsymbol{Q^3}$,
which is row 2 of $\boldsymbol{Q^2} \times$ row 1 of $\boldsymbol{Q}$, which is
${}_2Q^{(2,0)} \cdot Q^{(0,0)} + {}_2Q^{(2,1)} \cdot Q^{(1,0)} + {}_2Q^{(2,2)} \cdot Q^{(2,0)}$
$= (\frac{2}{3})(0) + (\frac{1}{3})(\frac{2}{3}) + (0)(0) = \frac{2}{9}$.

(b) $P(X_1 = 0) = P(X_1 = 0|X_0 = 0) \cdot P(X_0 = 0)$
$+ P(X_1 = 0|X_0 = 1) \cdot P(X_0 = 1) + P(X_1 = 0|X_0 = 2) \cdot P(X_0 = 2)$
$= Q^{(0,0)} \cdot P(X_0 = 0) + Q^{(1,0)} \cdot P(X_0 = 1) + Q^{(2,0)} \cdot P(X_0 = 2)$
$= (0)(\frac{1}{2}) + (\frac{2}{3})(\frac{1}{2}) + (0)(0) = \frac{1}{3}$.

$P(X_1 = 1) = P(X_1 = 1|X_0 = 0) \cdot P(X_0 = 0)$
$+ P(X_1 = 1|X_0 = 1) \cdot P(X_0 = 1) + P(X_1 = 1|X_0 = 2) \cdot P(X_0 = 2)$
$= Q^{(0,1)} \cdot P(X_0 = 0) + Q^{(1,1)} \cdot P(X_0 = 1) + Q^{(2,1)} \cdot P(X_0 = 2)$
$= (\frac{1}{3})(\frac{1}{2}) + (\frac{1}{3})(\frac{1}{2}) + (1)(0) = \frac{1}{3}$.

$P(X_1 = 2) = P(X_1 = 2|X_0 = 0) \cdot P(X_0 = 0)$
$+ P(X_1 = 2|X_0 = 1) \cdot P(X_0 = 1) + P(X_1 = 2|X_0 = 2) \cdot P(X_0 = 2)$
$= Q^{(0,2)} \cdot P(X_0 = 0) + Q^{(1,2)} \cdot P(X_0 = 1) + Q^{(2,2)} \cdot P(X_0 = 2)$
$= (\frac{2}{3})(\frac{1}{2}) + (0)(\frac{1}{2}) + (0)(0) = \frac{1}{3}$ (alternatively, $P(X_1 = 2) = 1 - P(X_1 = 0 \text{ or } 1))$.

(c) $Q^{(0,1)} > 0$, $Q^{(1,0)} > 0 \rightarrow 0, 1$ communicate.
$Q^{(2,1)} > 0$, ${}_2Q^{(1,2)} = Q^{(1,0)} \cdot Q^{(0,2)} > 0 \rightarrow 0, 2$ communicate.
Therefore all three states communicate

2. (d) W = number of transitions to reach state 2 given now in state 2.
$E_{i,j}$ = expected number of transitions to reach state j give now in state i.
$E_{2,2} = E[X]$ = expected number of transitions to return to state 2 given now in state 2.
We condition over the next transition:
$E_{2,2} = E[X] = E[X|\text{next transition is to 0}] \cdot Q^{(2,0)} + E[X|\text{next transition is to 1}] \cdot Q^{(2,1)}$
$\quad + E[X|\text{next transition is to 2}] \cdot Q^{(2,2)}$
$= (1 + E_{02})(0) + (1 + E_{1,2})(1) + (1 + E_{2,2})(0) = 1 + E_{1,2}$.

We set up a similar equation for $E_{1,2}$:
$E_{1,2} = (1 + E_{0,2}) \cdot Q^{(1,0)} + (1 + E_{1,2}) \cdot Q^{(1,1)} + (1) \cdot Q^{(1,2)}$
$= (1 + E_{0,2})(\frac{2}{3}) + (1 + E_{1,2}) \cdot (\frac{1}{3}) + (1)(0) = (1 + E_{0,2})(\frac{2}{3}) + (1 + E_{1,2}) \cdot (\frac{1}{3})$.
This equation becomes $\frac{2}{3}E_{1,2} = \frac{2}{3}E_{0,2} + 1$

We set up a similar equation for $E_{0,2}$:
$E_{0,2} = (1 + E_{0,2}) \cdot Q^{(0,0)} + (1 + E_{1,2}) \cdot Q^{(0,1)} + (1) \cdot Q^{(0,2)}$
$= (1 + E_{0,2})(0) + (1 + E_{1,2}) \cdot (\frac{1}{3}) + (1)(\frac{2}{3}) = (1 + E_{1,2}) \cdot (\frac{1}{3}) + \frac{2}{3}$.

The three equations are
$-E_{1,2} + E_{2,2} = 1$, $-\frac{2}{3}E_{0,2} + \frac{2}{3}E_{1,2} = 1$, $E_{0,2} - \frac{1}{3}E_{1,2} = 1$.
From the 2nd and 3rd equation we get $E_{0,2} = \frac{9}{4}$, $E_{1,2} = \frac{15}{4}$.
Then from equation 1 we get $E_{2,2} = \frac{19}{4}$.

(e) This probability is 0 since if $X_0 = 1$, then X_1 must be 0 or 1, so in order to avoid state 0 we must have $X_1 = 1$, then we must have $X_2 = 1$, but the one-step transition probability from state 1 to state 2 is 0.

3. (a) Let E_1 denote the expected number of years that the new resident in independent living will live in the home, and let E_2 denote the expected number of years that someone in the health care unit will live in the home. Then
$E_1 = (1 + E_1)(.6) + (1 + E_2)(.2) + (1)(.2)$ and
$E_2 = (1 + E_1)(.4) + (1 + E_2)(.4) + (1)(.2)$.
Solving these two equations results in $E_1 = 5$ and $E_2 = 5$.

(b) Let H_1 denote the expected number of years that the new resident will be living independently before leaving the home, and let $H_{2,1}$ denote the expected number of years that a resident in the health care unit will be living independently in the future before leaving the home. Then
$H_1 = (1 + H_1)(.6) + (1 + H_{21})(.2) + (1)(.2)$ and
$H_{21} = (H_1)(.4) + (H_{21})(.4) + (0)(.2)$.
Solving these two equations results in $H_1 = 3.75$ and $H_{21} = 2.5$.

3.(c) Let H_{12} denote the expected number of years that the new resident will be living in the health care unit before leaving the home, and let H_2 denote the expected number of years that a resident in the health care unit will be living in the health care unit at some point before leaving the home. Then
$H_{12} = (H_{12})(.6) + (H_2)(.2) + (0)(.2)$ and
$H_2 = (1 + H_{12})(.4) + (1 + H_2)(.4) + (1)(.2)$.
Solving these two equations results in $H_{12} = 1.25$ and $H_2 = 2.5$.

4.(a) $N_{11} =$ number of transition to return to state 1.
$E[N_{11}] = E[N_{11}|\text{next transition is to state 1}] \cdot P_{11} + E[N_{11}|\text{next transition is to state 2}] \cdot P_{12}$
$\rightarrow E_{11} = (1) \cdot \frac{1}{3} + (1 + E_{21}) \cdot \frac{2}{3} = \frac{2}{3}E_{21} + 1$, where E_{21} is the expected number of transitions from state 2 to get to state 1.
Similarly, $E_{21} = (1) \cdot \frac{1}{4} + (1 + E_{21}) \cdot \frac{3}{4}$, so that $E_{21} = 4$.
Then, $E_{11} = \frac{2}{3}(4) + 1 = \frac{11}{3}$.

(b) $E_{1,2,1} =$ expected number of future visits to state 2 before next return to state 1, given now in state 1
$E_{2,2,1} =$ expected number of future visits to state 2 before next visit to state 1, given now in state 2

$E_{1,2,1} = (0)(\frac{1}{3}) + (1 + E_{2,2,1})(\frac{2}{3})$
$E_{2,2,1} = (0)(\frac{1}{4}) + (1 + E_{2,2,1})(\frac{3}{4}) \rightarrow E_{2,2,1} = 3 \rightarrow E_{1,2,1} = \frac{8}{3}$.

(c) The eigenvalues are the solutions of $\begin{bmatrix} \frac{1}{3} - v & \frac{2}{3} \\ \frac{1}{4} & \frac{3}{4} - v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so that

$(\frac{1}{3} - v)(\frac{3}{4} - v) - \frac{2}{3} \cdot \frac{1}{4} = 0$. Solving for v results in $v = 1, \frac{1}{12}$, the eigenvalues.
The eigenvector for $v = 1$ is of the form $\begin{bmatrix} a \\ b \end{bmatrix}$, where $\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$.
Then $\frac{1}{3}a + \frac{2}{3}b = a \rightarrow a = b \rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the basic eigenvector for $v = 1$.
The eiegenvector for $v = \frac{1}{12}$ is of the form $\begin{bmatrix} a \\ b \end{bmatrix}$, where $\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \frac{1}{12}a \\ \frac{1}{12}b \end{bmatrix}$.
Then $\frac{1}{3}a + \frac{2}{3}b = \frac{1}{12}a \rightarrow b = -\frac{3}{8}a \rightarrow \begin{bmatrix} 1 \\ -\frac{3}{8} \end{bmatrix}$ is the basic eigenvector for $v = \frac{1}{12}$.
Then $\boldsymbol{D} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{12} \end{bmatrix}$ and $\boldsymbol{U} = \begin{bmatrix} 1 & 1 \\ 1 & -\frac{3}{8} \end{bmatrix}$.
Finally, $\boldsymbol{U}^{-1} = \frac{1}{(1)(-\frac{3}{8}) - (1)(1)} \cdot \begin{bmatrix} -\frac{3}{8} & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{11} & \frac{8}{11} \\ \frac{8}{11} & -\frac{8}{11} \end{bmatrix}$.

$\boldsymbol{Q = U \cdot D \cdot U^{-1}}$

$$\boldsymbol{Q^{(n)} = U \cdot D^n \cdot U^{-1}} = \begin{bmatrix} 1 & 1 \\ 1 & -\frac{3}{8} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{12^n} \end{bmatrix} \cdot \begin{bmatrix} \frac{3}{11} & \frac{8}{11} \\ \frac{8}{11} & -\frac{8}{11} \end{bmatrix} = \begin{bmatrix} \frac{3}{11} + \frac{8}{11\cdot 12^n} & \frac{8}{11} - \frac{8}{11\cdot 12^n} \\ \frac{3}{11} - \frac{3}{11\cdot 12^n} & \frac{8}{11} + \frac{3}{11\cdot 12^n} \end{bmatrix}$$

$P[X_{1000} = 1 \cap X_{1001} = 1 | X_0 = 1]$
$= P[X_{1001} = 1 | X_{1000} = 1 \cap X_0 = 1] \cdot P[X_{1000} = 1 | X_0 = 1]$
$= P[X_{1001} = 1 | X_{1000} = 1] \cdot P[X_{1000} = 1 | X_0 = 1]$
$= (\frac{1}{3})(\frac{3}{11} + \frac{8}{11\cdot 12^{1000}}) = \frac{1}{11}$.

5. (a) There is a payment of 1000 at time 1 with probability $Q_0{}^{(1,2)}$.
There is a payment of 1000 at time 2 with probability ${}_2Q_0^{(1,2)}$.
There is a payment of 1000 at time 3 with probability ${}_3Q_0^{(1,2)}$.
$Q_0{}^{(1,2)} = .25$.
${}_2Q_0{}^{(1,2)} = Q_0{}^{(1,2)} \cdot Q_1^{(2,2)} + Q_0{}^{(1,1)} \cdot Q_1^{(1,2)} = (.25)(.5) + (.5)(.2) = .225$.
Alternatively, ${}_2Q_0{}^{(1,2)}$ is the $(1,2)$-entry in $\boldsymbol{{}_2Q = Q_0 \times Q_1}$.

${}_3Q_0{}^{(1,2)}$ is the $(1,2)$-entry in $\boldsymbol{{}_3Q = {}_2Q_0 \times Q_1}$.

$$\boldsymbol{{}_2Q_0} = \begin{bmatrix} .5 & .25 & .25 \\ - & - & - \\ - & - & - \end{bmatrix} \times \begin{bmatrix} .6 & .2 & .2 \\ .4 & .5 & .1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} .4 & .225 & .375 \\ - & - & - \\ - & - & - \end{bmatrix}$$

$$\boldsymbol{{}_3Q_0} = \begin{bmatrix} .4 & .225 & .375 \\ - & - & - \\ - & - & - \end{bmatrix} \times \begin{bmatrix} .8 & .1 & .1 \\ .25 & .75 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} .37625 & .20875 & .415 \\ - & - & - \\ - & - & - \end{bmatrix}$$

${}_3Q_0{}^{(1,2)} = .20875$. This can also be found by identifying paths from state 1 at time 0 to state 2 at time 3.
The expected amount of scholarship received by a student is
$1000(.25 + .225 + .20875) = 683.75$.

(b) The probability of a new student getting this award at the end of the 2nd year is
$Q_0^{(1,1)} \cdot Q_1^{(1,2)} = (.5)(.2) = .1$
The probability of a new student getting this award at the end of the 3rd year is
${}_2Q_0^{(1,1)} \cdot Q_2^{(1,2)} = (.4)(.1) = .04$.
The expected amount that will be paid for this award to a new student is $1000(.1 + .04) = 140$.

(c) The probability that a new student will never be named to the dean's honour list is
$Q_0^{(1,3)} + Q_0^{(1,1)} \cdot Q_1^{(1,3)} + Q_0^{(1,1)} \cdot Q_1^{(1,1)} \cdot Q_2^{(1,3)} + Q_0^{(1,1)} \cdot Q_1^{(1,1)} \cdot Q_2^{(1,1)}$
$= .25 + (.5)(.2) + (.5)(.6)(.1) + (.5)(.6)(.8) = .62$.
The probability of being named at least once is .38 .
Alternatively, the probability is
$Q_0^{(1,2)} + Q_0^{(1,1)} \cdot Q_1^{(1,2)} + Q_0^{(1,1)} \cdot Q_1^{(1,1)} \cdot Q_2^{(1,2)}$
$= .25 + (.5)(.2) + (.5)(.6)(.1) = .25 + .1 + .03 = .38$.

6.(a) ${}_2\boldsymbol{Q} = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \to {}_2Q^{(2,0)} = 0$, ${}_3\boldsymbol{Q} = \begin{bmatrix} \frac{1}{8} & \frac{1}{2} & \frac{3}{8} \\ \frac{1}{2} & \frac{3}{8} & \frac{1}{8} \\ \frac{3}{4} & \frac{1}{4} & 0 \end{bmatrix} \to {}_3Q^{(2,0)} = \frac{3}{4}$.

(b) $\boldsymbol{\alpha} = [\,\frac{1}{3}\ \ \frac{1}{3}\ \ \frac{1}{3}\,] \to$ unconditional distribution of X_1 is $\boldsymbol{\alpha} \cdot \boldsymbol{Q} = [\,\frac{1}{2}\ \ \frac{1}{3}\ \ \frac{1}{6}\,]$,
unconditional distribution of X_2 is $\boldsymbol{\alpha} \cdot {}_2\boldsymbol{Q} = [\,\frac{1}{3}\ \ \frac{5}{12}\ \ \frac{1}{4}\,]$.

(c) We convert state 2 to an absorbing state, so the transition matrix becomes
$\boldsymbol{Q'} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then ${}_2\boldsymbol{Q'} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 1 \end{bmatrix}$

and $\quad {}_3\boldsymbol{Q'} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{8} & \frac{1}{4} & \frac{5}{8} \\ \frac{1}{4} & \frac{3}{8} & \frac{3}{8} \\ 0 & 0 & 1 \end{bmatrix}$

Then $P[X_3 = 1,\ X_2 \neq 2,\ X_1 \neq 2\,|X_0 = 1] = ({}_3Q^{(1,1)})' = \frac{3}{8}$.

7. (a) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(b) $P_2 =$ prob. reaching 3 before 0 from state 2 , $P_1 =$ prob. reaching 3 before 0 from state 1.
$P_2 = P_1 \cdot \frac{2}{3} + (1)(\frac{1}{3})$ and $P_1 = (0)(\frac{1}{3}) + P_2 \cdot \frac{2}{3} \to P_2 = \frac{3}{5}$.

(c) $E_2 =$ expected number of bets until reaching 0 or 3 from state 2,
$E_1 =$ expected number of bets until reaching 0 or 3 from state 1.
$E_2 = (1)(\frac{1}{3}) + (1 + E_1)(\frac{2}{3})$ and $E_1 = (1 + E_2)(\frac{2}{3}) + (1)(\frac{1}{3}) \to E_2 = 3$.

(d) $R_{2,2} =$ expected number of times in the future the gambler will have 2 given he now has 2
$R_{1,2} =$ expected number of times in the future the gambler will have 2 given he now has 1.
$R_{2,2} = (0)(\frac{1}{3}) + R_{1,2}(\frac{2}{3})$ and $R_{1,2} = (1 + R_{2,2})(\frac{2}{3}) + (0)(\frac{1}{3}) \to R_{2,2} = .8$

8.(a) We convert state H to an absorbing state. The transition matrices become

$$\boldsymbol{Q_0} = \begin{array}{cccc} & L & M & H \\ L & .6 & .3 & .1 \\ M & .3 & .4 & .3 \\ H & 0 & 0 & 1 \end{array}$$ for the first year,

and $$\boldsymbol{Q_n} = \begin{array}{cccc} & L & M & H \\ L & .8 & .1 & .1 \\ M & .2 & .6 & .2 \\ H & 0 & 0 & 1 \end{array}$$ for the second and subsequent years ($n \geq 1$).

Then $${}_3\boldsymbol{Q} = \boldsymbol{Q_0} \times \boldsymbol{Q_1} \times \boldsymbol{Q_2} = \begin{array}{cccc} & L & M & H \\ L & .48 & .198 & .322 \\ M & .31 & .194 & .496 \\ H & 0 & 0 & 1 \end{array}$$

Then the probability in question is ${}_3Q_0^{(L,L)} + {}_3Q_0^{(L,M)} = .48 + .198 = .678$
(the probability of being rated L or M at the end of 3 years).

(b) For a policyholder rated Low risk at time 0, expected first year claim is 100.
The rating probabilities at the start of the second year for this policyholder are
L: $Q_0^{(L,L)} = .6$, M: $Q_0^{(L,M)} = .3$, H: $Q_0^{(L,H)} = .1$.
The expected claim in the 2nd year is $(.6)(100) + (.3)(200) + (.1)(1000) = 220$.
The two-step transition probabilities from time 0 are in the matrix

$${}_2\boldsymbol{Q} = \begin{bmatrix} .6 & .3 & .1 \\ - & - & - \\ - & - & - \end{bmatrix} \times \begin{bmatrix} .8 & .1 & .1 \\ .2 & .6 & .2 \\ 0 & .1 & .9 \end{bmatrix} = \begin{bmatrix} .54 & .24 & .22 \\ - & - & - \\ - & - & - \end{bmatrix}.$$

The rating probabilities at the start of the third year for this policyholder are
L: ${}_2Q_0^{(L,L)} = .54$, M: $2Q_0^{(L,M)} = .25$, H: $2Q_0^{(L,H)} = .21$.
The expected claim in the 3rd year is $(.54)(100) + (.25)(200) + (.21)(1000) = 314$.
Three-year total expected claim is 634 .

(c)(i) Expected value of premiums is
$125(1 + Q_0^{(L,L)} + {}_2Q_0^{(L,L)}) + 250(Q_0^{(L,M)} + {}_2Q_0^{(L,M)}) + K(Q_0^{(L,H)} + {}_2Q_0^{(L,H)})$
$= 125(1 + .6 + .54) + 250(.3 + .24) + K(.1 + .22) = 642 \rightarrow K = 748.4375$

(ii) At time 1 for a Medium risk driver, the expected value of future claims is
$200 + 100Q_1^{(M,L)} + 200Q_1^{(M,M)} + 1000Q_1^{(M,H)}$
$= 200 + 100(.2) + 200(.6) + 1000(.2) = 540$.
The expected value of future premiums is
$250 + 125Q_1^{(M,L)} + 250Q_1^{(M,M)} + 748.4Q_1^{(M,H)}$
$= 250 + 125(.2) + 250(.6) + 748.4(.2) = 574.7$.
The reserve is APV benefit $-$ APV premium $= 540 - 574.7 = -34.7$.

(d) There are two possible rating changes at the end of the first year that will results in fees:
L to M, \$50 with prob. $Q_0^{(L,M)} = .3$ and L to H, \$200 with prob. $Q_0^{(L,H)} = .1$.
There are three possible rating changes at the end of the second year that will result in fees:
L to M, \$50 with prob. $Q_0^{(L,L)} \cdot Q_1^{(L,M)} = (.6)(.1) = .06$,
L to H, \$200 with prob. $Q_0^{(L,L)} \cdot Q_1^{(L,H)} = (.6)(.1) = .06$, and
M to H, \$100 with prob. $Q_0^{(L,M)} \cdot Q_1^{(M,H)} = (.3)(.2) = .06$.
The expected rating change fee for the first two years is
$(50)(.3) + (200)(.1) + (50)(.06) + (200)(.06) + (100)(.06) = 56$.