ACT451F - TERM TEST 2 - NOVEMBER 18, 2004

Print your name and student number clearly on each page

Do all work on this question paper, with work for question 1 on this page (use both sides if necessary), work for question 2 on the next page, etc. All calculations should be done to at least 6 digit accuracy. Answers are expected to have at least 4 digit accuracy.
THIS IS A CLOSED BOOK TEST - NO BOOKS OR NOTES ARE ALLOWED.
A calculator may be used

1. A loss random variable X has an exponential distribution with a mean of λ.
λ is also a random variable, with an inverse gamma distribution with density function $\pi(\lambda) = \frac{600^4 \cdot e^{-600/\lambda}}{6\lambda^5}$.

4 (a) Find the mean and variance of the unconditional distribution of X.

4 (b) Show that the unconditional distribution of X has a Pareto distribution, and determine the α and θ parameter values.
(You may use the fact that $\int_0^\infty \frac{e^{-c/t}}{t^n}\, dt = \frac{(n-2)!}{c^{n-1}}$)

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Solution: λ has parameters $\alpha = 4$ and $\theta = 600$.

(a) $E[X] = E[\,E[X|\lambda]\,] = E[\lambda] = \frac{\theta}{\alpha-1} = 200$.
$Var[X] = Var[\,E[X|\lambda]\,] + E[\,Var[X|\lambda]\,]$
$= Var[\lambda] + E[\lambda^2] = [\frac{\theta^2}{(\alpha-1)(\alpha-2)} - (\frac{\theta}{\alpha-1})^2] + \frac{\theta^2}{(\alpha-1)(\alpha-2)} = [\frac{600^2}{(3)(2)} - 200^2] + \frac{600^2}{(3)(2)}$
$= 80,000$

(b) $f_X(x) = \int_0^\infty f(x|\lambda) \cdot \pi(\lambda)\, d\lambda = \int_0^\infty \frac{1}{\lambda} e^{-x/\lambda} \cdot \frac{600^4 \cdot e^{-600/\lambda}}{6\lambda^5}\, d\lambda = \frac{600^4}{6} \int_0^\infty \frac{e^{-(600+x)/\lambda}}{\lambda^6}\, d\lambda$
$= \frac{600^4}{6} \cdot \frac{4!}{(x+600)^5} = \frac{4 \cdot 600^4}{(x+600)^5}$. This is the pdf of a Pareto distribution with $\alpha = 4$ and $\theta = 600$.

2. A compound distribution S has frequency distribution N which is Poisson with a mean of 1. The severity distribution X is a discrete random variable, with
$P[X=k]=\frac{1}{2^k}$ for $k=1,2,3,...$

4 (a) Find $P[S>2]$

4 (b) Find $P[S=3]$ two ways

(i) Using a combinatorial approach by considering all combinations of N and X which result in $S=3$.

(ii) Using the recursive approach that applies if the frequency is in the $(a,b,0)$ class.

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Solution: (a) $P[S>2]=1-P[S=0,1,2]$

$P[S=0]=P[N=0]=e^{-1}$, $P[S=1]=P[N=1]\cdot P[X=1]=e^{-1}\cdot\frac{1}{2}$

$P[S=2]=P[N=1]\cdot P[X=2]+P[N=2]\cdot(P[X=1])^2$

$=e^{-1}\cdot\frac{1}{4}+\frac{e^{-1}}{2}\cdot(\frac{1}{2})^2=\frac{3}{8}e^{-1}$.

Then $P[S=0,1,2]=\frac{15}{8}e^{-1}$ and $P[S>2]=1-\frac{15}{8}e^{-1}$

(b)(i) $P[S=3]=P[N=1]\cdot P[X=3]+P[N=2]\cdot 2\cdot P[X=1]\cdot P[X=2]$
$+P[N=3]\cdot(P[X=1])^3$

$=e^{-1}\cdot\frac{1}{8}+\frac{e^{-1}}{2}\cdot 2\cdot(\frac{1}{2})(\frac{1}{4})+\frac{e^{-1}}{6}\cdot(\frac{1}{2})^3=\frac{13}{48}e^{-1}$

(ii) $g_k=P[S=k]=\frac{1}{1-af_0}\cdot\sum_{j=1}^{k}(a+\frac{bj}{k})f_j\cdot g_{k-j}$, $k=1,2,3,...$

For the Poisson $(a,b,0)$ dist. $a=0$ and $b=\lambda$, so

$g_k=P[S=k]=\sum_{j=1}^{k}\frac{\lambda j}{k}\cdot f_j\cdot g_{k-j}$, $k=1,2,3,...$

We start with $g_0=P[S=0]=e^{-1}$ (from (a) above)

and $f_j=P[X=j]=\frac{1}{2^j}$ $j=1,2,...$

Then $g_1=f_1\cdot g_0=\frac{1}{2}e^{-1}$,

$g_2=\frac{1}{2}\cdot f_1\cdot g_1+\frac{(1)(2)}{2}\cdot f_2\cdot g_0=\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}e^{-1}+\frac{1}{4}\cdot e^{-1}=\frac{3}{8}e^{-1}$,

$g_3=\frac{1}{3}\cdot f_1\cdot g_2+\frac{(1)(2)}{3}\cdot f_2\cdot g_1+\frac{(1)(3)}{3}\cdot f_3\cdot g_0$

$=\frac{1}{3}\cdot\frac{1}{2}\cdot\frac{3}{8}e^{-1}+\frac{2}{3}\cdot\frac{1}{4}\cdot\frac{1}{2}e^{-1}+\frac{1}{8}\cdot e^{-1}=\frac{13}{48}e^{-1}=g_3=P[S=3]$

3. The frequency distribution N is binomial with parameters 10 and $.1$, so $P[N=k] = \binom{10}{k}(.1)^k(.9)^{10-k}$. The severity distribution X is uniform on the interval $[0,100]$. A deductible of 25 is applied to each individual loss X. Let N^* denote the number of losses above the deductible. Use the probability generating function of the binomial distribution to show that N^* has a binomial distribution with parameters 10 and $.075$. The probability generating function of a binomial with parameters n and p is $P(t) = [1-p+pt]^n$.
(Hint: Recall that N^* can be formulated as a compound distribution using an indicator random variable.)

Solution: $I = \begin{cases} 0 & X \leq 25 \text{ , prob. .25} \\ 1 & X > 25 \text{ , prob. .75} \end{cases}$. $N^* = I_1 + I_2 + \cdots + I_N$

$P_I(t) = .25 + .75t$

$P_{N^*}(t) = P_N(P_I(t)) = P_N[.25 + .75t] = [1 - .1 + (.1)(.25 + .75t)]^{10}$
$= [1 - .075 + .075t]^{10}$, which is the pgf for the binomial dist with paramters 10, .075 .