Continuous Mixture of Poisson over Gamma

Assume that λ has a gamma distribution with parameters α and θ, so the pdf is

$$\pi(\lambda) = \frac{\lambda^{\alpha-1} \cdot e^{-\lambda/\theta}}{\theta^{\alpha} \cdot \Gamma(\alpha)} \ .$$

Assume that the conditional distribution of X given λ is Poisson with mean λ, so the probability function of X given λ is $P[X=k|\lambda] = \frac{e^{-\lambda}\lambda^k}{k!}$.

The unconditional probability function of X is

$$P[X=k] = \int_0^{\infty} P[X=k|\lambda] \cdot \pi(\lambda) d\lambda = \int_0^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} \cdot \frac{\lambda^{\alpha-1} \cdot e^{-\lambda/\theta}}{\theta^{\alpha} \cdot \Gamma(\alpha)} \, d\lambda$$

$$= \int_0^{\infty} \frac{\lambda^{\alpha+k-1} \cdot e^{-\lambda/[\frac{\theta}{1+\theta}]}}{k! \cdot \theta^{\alpha} \cdot \Gamma(\alpha)} \, d\lambda = \frac{[\frac{\theta}{1+\theta}]^{\alpha+k} \cdot \Gamma(\alpha+k)}{k! \cdot \theta^{\alpha} \cdot \Gamma(\alpha)} \cdot \int_0^{\infty} \frac{\lambda^{\alpha+k-1} \cdot e^{-\lambda/[\frac{\theta}{1+\theta}]}}{[\frac{\theta}{1+\theta}]^{\alpha+k} \cdot \Gamma(\alpha+k)} \, d\lambda$$

$= \frac{[\frac{\theta}{1+\theta}]^{\alpha+k} \cdot \Gamma(\alpha+k)}{k! \cdot \theta^{\alpha} \cdot \Gamma(\alpha)}$ (this is because $\frac{\lambda^{\alpha+k-1} \cdot e^{-\lambda/[\frac{\theta}{1+\theta}]}}{[\frac{\theta}{1+\theta}]^{\alpha+k} \cdot \Gamma(\alpha+k)}$ is the pdf for a gamma distribution with parameters $\alpha' = \alpha + k$ and $\theta' = \frac{\theta}{1+\theta}$, so its integral is 1).

This shows that the unconditional probability function of X is

$$P[X=k] = \frac{[\frac{\theta}{1+\theta}]^{\alpha+k} \cdot \Gamma(\alpha+k)}{k! \cdot \theta^{\alpha} \cdot \Gamma(\alpha)} = \frac{\Gamma(\alpha+k)}{k! \cdot \Gamma(\alpha)} \left(\frac{1}{1+\theta}\right)^{\alpha} \left(\frac{\theta}{1+\theta}\right)^{k}.$$

The Negative Binomial Distribution

The following is from LM-117 in Volume II of the study guide.

The probability function of the negative binomial random variable N with parameters $r > 0$ and $\beta > 0$ is $p_k = P[N=k] = \binom{k+r-1}{k} \left(\frac{1}{1+\beta}\right)^r \left(\frac{\beta}{1+\beta}\right)^k$, $k = 0, 1, 2, ...,$

where $\binom{x}{k} = \frac{(x)(x-1)\cdots(x-k+1)}{k!} = \frac{\Gamma(x+1)}{\Gamma(k+1)\Gamma(x-k+1)}$ for integer $k \geq 0$, and any real x .

The probability generating function of N is $P_N(t) = \frac{1}{[1-\beta(t-1)]^r}$.

The moment generating function of N is $M_N(t) = \frac{1}{[1-\beta(e^t-1)]^r}$.

The mean and variance of N are $E[N] = r\beta \ < \ Var[N] = r\beta(1+\beta)$.

If $r = 1$, this distribution is referred to as the **geometric distribution**, and $p_k = \frac{\beta^k}{(1+\beta)^{k+1}}$.

Note that it is always true for any x that $\binom{x}{0} = 1$ and $\binom{x}{1} = x$.

For instance, if $r = 2$ and $\beta = 2$, then $P[N=0] = \binom{0+2-1}{0} \left(\frac{1}{1+2}\right)^2 \left(\frac{2}{1+2}\right)^0 = (1)(\frac{1}{9})(1) = \frac{1}{9}$ and $P[N=1] = \binom{1+2-1}{1} \left(\frac{1}{1+2}\right)^2 \left(\frac{2}{1+2}\right)^1 = (2)(\frac{1}{9})(\frac{2}{3}) = \frac{4}{27}$ and

$$P[N=2] = \binom{2+2-1}{2} \left(\frac{1}{1+2}\right)^2 \left(\frac{2}{1+2}\right)^2 = (3)(\tfrac{1}{9})(\tfrac{4}{9}) = \frac{4}{27} \ .$$

The example in class was the number of failures before the 3rd success, where success is defined as tossing a "1" and failure is defined as tossing a "2,3,4,5,6" when tossing a fair die. The probability of success is $\frac{1}{6}$ and the probability of failure is $\frac{5}{6}$ on any particular toss. In order to have k failures before the 3rd success, it must be true that there are k failures and 2 success in the first $k+2$ tosses and then the 3rd succes is on the $k+3$-rd toss. The probability of this is (Prob. 2 successes in first $k+2$ tosses) $\times$ (Prob. of success on $k+3$-rd toss) .
The number of successes in the first $k+2$ tosses has a binomial dist. with prob of succes $\frac{1}{6}$, so Prob. 2 successes in first $k+2$ tosses $= \binom{k+2}{k}(\frac{1}{6})^2(\frac{5}{6})^k$.
Then, Prob of exactly k failures before the 3rd success is
$\binom{k+2}{k}(\frac{1}{6})^2(\frac{5}{6})^k \times \frac{1}{6} = \binom{k+2}{k}(\frac{1}{6})^3(\frac{5}{6})^k$.

The binomial coefficient $\binom{k+2}{k}$ is equal to $\frac{(k+2)!}{k!\cdot 2!} = \frac{\Gamma(k+3)}{\Gamma(k+1)\cdot\Gamma(3)}$, so that the probability function for X, the number of failures before the 3rd success is $\frac{\Gamma(k+3)}{\Gamma(k+1)\cdot\Gamma(3)} \cdot (\frac{1}{6})^3(\frac{5}{6})^k$.
This can be written as $\frac{\Gamma(\alpha+k)}{\Gamma(k+1)\cdot\Gamma(\alpha)}(\frac{1}{1+\theta})^\alpha(\frac{\theta}{1+\theta})^k$, where $\alpha = 3$ and $\theta = 5$, so the probability of success in any particular trial in this situation is $\frac{1}{1+\theta} = \frac{1}{6}$.
The general form of the negative binomial distribution has probability function
$P[N=k] = \binom{k+r-1}{k}\left(\frac{1}{1+\beta}\right)^r\left(\frac{\beta}{1+\beta}\right)^k = P[N=k] = \frac{\Gamma(k+r)}{\Gamma(k+1)\cdot\Gamma(r)}\left(\frac{1}{1+\beta}\right)^r\left(\frac{\beta}{1+\beta}\right)^k$.
So we see that for the die toss situation the number of failures before the 3rd success has a negative binomial distribution with $r = 3$ and $\theta = 5$.