ACT348F - TERM TEST 1 - OCTOBER 17, 2001

Print your name and student number clearly on each page

Do all work on this question paper, with work for question 1 on this page (use both sides if necessary), work for question 2 on the next page, etc. All calculations should be done to at least 6 digit accuracy. Answers are expected to have at least 4 digit accuracy.

THIS IS A CLOSED BOOK TEST - NO BOOKS OR NOTES ARE ALLOWED.

A calculator may be used

1. (a) Express ${}_t\overline{V}(\overline{A}_{x:\overline{n}|})$ in each of the following forms:

(i) prospective, (ii) retrospective, (iii) premium difference, (iv) paid-up insurance
(v) annuity-only, (vi) insurance only, (vii) premium only

(b) Formulate the prospective and retrospective form of the reserve ${}^k_tV_{x:\overline{n}|}$.
Starting with one of these formulations, show that if $t < k$,
then ${}^k_tV_{x:\overline{n}|} = \frac{{}_kP_{x:\overline{n}|} - P_{x:\overline{t}|}^{\;\,1}}{P_{x:\overline{t}|}^{\;\,1}}$.

2.(a) With $i = .1$ and assuming DeMoivre's Law with $\omega = 100$ $\left({}_tp_x = \frac{100-x-t}{100-x}\right)$ find ${}_{20}\overline{V}(\overline{A}_{50})$.

(b) With $\delta = .1$ (force of interest) and assuming $\mu(y) = .02$ for all y (constant force of mortality at all ages), find ${}_{20}\overline{V}(\overline{A}_{50})$.

3.(a) Using the attached table (at $i = .06$) and assuming UDD, find ${}_{20}V(\overline{A}_{50})$.
(b) Assuming UDD, derive the relationship
${}_tV^{(m)}(\overline{A}_x) = {}_tV(\overline{A}_x) + \beta(m) \cdot P^{(m)}(\overline{A}_x) \cdot {}_tV_x$
BONUS: It can be shown that $\frac{d}{dy}\overline{a}_y = \int_0^\infty v^t \cdot {}_tp_y \cdot (\mu(y) - \mu(y+t))\, dt$.
Using this, prove that if $\mu(z)$ is an increasing function of z (force of mortality increases with age) then $\frac{d}{dt}\,{}_t\overline{V}(\overline{A}_x) > 0$ (fully continuous whole life reserve increases with t).

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1. (a) (i) prospective: $\overline{A}_{x+t:\overline{n-t}|} - \overline{P}(\overline{A}_{x:\overline{n}|}) \cdot \overline{a}_{x+t:\overline{n-t}|}$,

(ii) retrospective: $\overline{P}(\overline{A}_{x:\overline{n}|}) \cdot \overline{s}_{x:\overline{t}|} - \frac{\overline{A}^{\ 1}_{x:\overline{t}|}}{v^t\ {}_tp_x}$

(iii) premium difference: $[\ \overline{P}(\overline{A}_{x+t:\overline{n-t}|}) - \ \overline{P}(\overline{A}_{x:\overline{n}|})] \cdot \overline{a}_{x+t:\overline{n-t}|}$,

(iv) paid-up insurance: $[1 - \frac{\overline{P}(\overline{A}_{x:\overline{n}|})}{\overline{P}(\overline{A}_{x+t:\overline{n-t}|})}] \cdot \overline{A}_{x+t:\overline{n-t}|}$,

(v) annuity only: $1 - \frac{\overline{a}_{x+t:\overline{n-t}|}}{\overline{a}_{x:\overline{n}|}}$, (vi) insurance only: $\frac{\overline{A}_{x+t:\overline{n-t}|} - \overline{A}_{x:\overline{n}|}}{1 - \overline{A}_{x:\overline{n}|}}$,

(vii) premium only: $\frac{\overline{P}(\overline{A}_{x+t:\overline{n-t}|}) - \overline{P}(\overline{A}_{x:\overline{n}|})}{\overline{P}(\overline{A}_{x+t:\overline{n-t}|}) + \delta}$

(b) Prospective: ${}^k_tV_{x:\overline{n}|} = \begin{cases} A_{x+t:\overline{n-t}|} - {}_kP_{x:\overline{n}|} \cdot \ddot{a}_{x+t:\overline{n-t}|} & t < k \\ A_{x+t:\overline{n-t}|} & t \geq k \end{cases}$

Retrospective: ${}^k_tV_{x:\overline{n}|} = \begin{cases} {}_kP_{x:\overline{n}|} \cdot \ddot{s}_{x:\overline{t}|} - \frac{A^{\ 1}_{x:\overline{t}|}}{v^t\ {}_tp_x} & t < k \\ {}_kP_{x:\overline{n}|} \cdot \ddot{s}_{x:\overline{k}|} \cdot \frac{1}{v^{t-k}\ {}_{t-k}p_{x+k}} - \frac{A^{\ 1}_{x:\overline{t}|}}{v^t\ {}_tp_x} & k \leq t (\leq n) \end{cases}$

From the retrospective with $t < k$,

${}^k_tV_{x:\overline{n}|} = {}_kP_{x:\overline{n}|} \cdot \ddot{s}_{x:\overline{t}|} - \frac{A^{\ 1}_{x:\overline{t}|}}{v^t\ {}_tp_x} = \frac{{}_kP_{x:\overline{n}|}}{P_{x:\overline{t}|}^{\ \ 1}} - \frac{A^{\ 1}_{x:\overline{t}|}/\ddot{a}_{x:\overline{t}|}}{v^t\ {}_tp_x/\ddot{a}_{x:\overline{t}|}} = \frac{{}_kP_{x:\overline{n}|} - P^{\ 1}_{x:\overline{t}|}}{P_{x:\overline{t}|}^{\ \ 1}}$

2.(a) ${}_{20}\overline{V}(\overline{A}_{50}) = \frac{\overline{A}_{70} - \overline{A}_{50}}{1 - \overline{A}_{50}}$. Under DeMoivre's Law $\overline{A}_x = \frac{1}{\omega - x} \cdot \overline{a}_{\overline{\omega - x}|}$.

$\overline{A}_{70} = \frac{1}{100-70} \cdot \overline{a}_{\overline{100-70}|} = \frac{1}{30} \cdot \overline{a}_{\overline{30}|} = \frac{1}{30} \cdot \frac{1-v^{30}}{\delta} = .3297$.

$\overline{A}_{50} = \frac{1}{100-50} \cdot \overline{a}_{\overline{100-50}|} = \frac{1}{50} \cdot \overline{a}_{\overline{50}|} = \frac{1}{50} \cdot \frac{1-v^{50}}{\delta} = .2081$.

${}_{20}\overline{V}(\overline{A}_{50}) = \frac{\overline{A}_{70} - \overline{A}_{50}}{1 - \overline{A}_{50}} = \frac{.3297 - .2081}{1 - .2081} = .1536$.

(b) ${}_{20}\overline{V}(\overline{A}_{50}) = \frac{\overline{A}_{70} - \overline{A}_{50}}{1 - \overline{A}_{50}}$. Under constant force of mortality μ and constant force of interest δ, $\overline{A}_x = \int_0^\infty e^{-\delta t} \cdot {}_tp_x \cdot \mu(x+t)\, dt = \int_0^\infty e^{-\delta t} \cdot e^{-\mu t} \cdot \mu\, dt = \frac{\mu}{\delta + \mu}$.

Then, $\overline{A}_{70} = \overline{A}_{50} = \frac{\mu}{\delta + \mu} \rightarrow {}_{20}\overline{V}(\overline{A}_{50}) = 0$.

3.(a) ${}_{20}V(\overline{A}_{50}) = \overline{A}_{70} - P(\overline{A}_{50}) \cdot \ddot{a}_{70} = \frac{i}{\delta} \cdot {}_{20}V_{50} = \frac{.06}{ln\, 1.06} \cdot [1 - \frac{\ddot{a}_{70}}{\ddot{a}_{50}}]$

$= .3646$.

(b) ${}_tV^{(m)}(\overline{A}_x) = \overline{A}_{x+t} - P^{(m)}(\overline{A}_x)\ddot{a}^{(m)}_{x+t}$

$= \overline{A}_{x+t} - P(\overline{A}_x)\ddot{a}_{x+t} + P(\overline{A}_x)\ddot{a}_{x+t} - P^{(m)}(\overline{A}_x)\ddot{a}^{(m)}_{x+t}$

$= {}_tV(\overline{A}_x) + P^{(m)}(\overline{A}_x)[\frac{P(\overline{A}_x)}{P^{(m)}(\overline{A}_x)}\ddot{a}_{x+t} - \ddot{a}^{(m)}_{x+t}]$

$= {}_tV(\overline{A}_x) + P^{(m)}(\overline{A}_x)[\frac{\overline{A}_x/\ddot{a}_x)}{\overline{A}_x/\ddot{a}^{(m)}_x}\ddot{a}_{x+t} - \ddot{a}^{(m)}_{x+t}]$

$= {}_tV(\overline{A}_x) + P^{(m)}(\overline{A}_x)[\frac{\ddot{a}^{(m)}_x}{\ddot{a}_x}\ddot{a}_{x+t} - \ddot{a}^{(m)}_{x+t}]$

$= {}_tV(\overline{A}_x) + P^{(m)}(\overline{A}_x)[\frac{\ddot{a}^{(m)}_x \cdot \ddot{a}_{x+t} - \ddot{a}_x \cdot \ddot{a}^{(m)}_{x+t}}{\ddot{a}_x}]$

$= {}_tV(\overline{A}_x) + P^{(m)}(\overline{A}_x)[\frac{(\alpha(m)\ddot{a}_x - \beta(m)) \cdot \ddot{a}_{x+t} - \ddot{a}_x \cdot (\alpha(m)\ddot{a}_{x+t} - \beta(m))}{\ddot{a}_x}]$

$= {}_tV(\overline{A}_x) + P^{(m)}(\overline{A}_x)\beta(m)[1 - \frac{\ddot{a}_{x+t}}{\ddot{a}_x}] = {}_tV(\overline{A}_x) + P^{(m)}(\overline{A}_x)\beta(m){}_tV_x$

BONUS: $\frac{d}{dt}\,{}_t\overline{V}(\overline{A}_x) = \frac{d}{dt}(1 - \frac{\overline{a}_{x+t}}{\overline{a}_x}) = -\frac{1}{\overline{a}_x} \cdot \frac{d}{dt}\,\overline{a}_{x+t} > 0$, since $\frac{d}{dt}\,\overline{a}_{x+t} < 0$.