Davis & Yau (2011)

true model $X_t = \cancel{\mu} + \sum_{j=0}^{\infty} \psi_j Z_{t-j}$ $t = 1, 2, \dots, n, \dots$ Z_j iid $(0, \sigma^2)$

'general t.s.' $\mu = 0$ $\psi_0 = 1$ $\sum_j \psi_j < \infty$ $\psi_j = \psi_j(\theta)$ $\theta \in \mathbb{R}^m$

model for fitting assume $f(x_t, x_{t+k})$ $\Delta_\theta(k) = \gamma_\theta^2(0) - \gamma_\theta^2(k)$

$$= \frac{1}{2\pi\sigma^2\sqrt{\Delta_\theta(k)}} e^{-\frac{1}{2\pi\sigma^2\Delta_\theta(k)}\left\{(x_t^2 + x_{t+k}^2)\gamma_\theta(0) - 2x_t x_{t+k}\gamma_\theta(k)\right\}}$$

$\sigma^2\gamma_\theta(k) = E(X_{t+k} X_t)$

pairwise log-lik $-\frac{1}{2}\sum_{k=1}^{n-p}\sum_{t=1}^{n-k}\left[\left\{ \quad \right\} / \sigma^2\Delta_\theta(k) + \log \sigma^4 \Delta_\theta(k)\right]$

((log))

consecutive PL $-\frac{1}{2}\sum_{t=1}^{n-1} f(x_t, x_{t+1}; \theta)$

s.m.
$\sum|\gamma_\theta(k)|<\infty$

$\hat{\eta}_{cl} = (\hat{\theta}_{cl}, \hat{\sigma}^2_{cl})$ $\eta_0 = (\theta_0, \sigma_0^2)$ $X = (X_1, X_2, \dots)$

$(n=1, m \to \infty)$ (old not^n)

l.m.
$\gamma_\theta(k) \sim \lambda |k|^{2d-1}$
$d < \frac{1}{2}$

Consec. PL

i) short memory, or
long memory, $d < \frac{1}{4}$ $\sqrt{n}(\hat{\eta}_{cl} - \eta_0) \xrightarrow{d} N(0, \_\_)$

God. info.

ii) " " $d = \frac{1}{4}$ $\sqrt{n/\log n}(\hat{\eta}_{cl} - \eta_0) \xrightarrow{d} N(0, \text{mess})$

iii) " " $d > \frac{1}{4}$ $n^{1-2d}(\hat{\eta}_{cl} - \eta_0) \xrightarrow{d}$ something but not N

All pairs: short mem. $\hat{\eta}_{cl} \xrightarrow{p} \eta_0$
if not ??

$\hat{\theta}_{cl}$ eff't

Numerical work
AR(1) $X_t = \phi X_{t-1} + Z_t$ ← CPL ✓ (cons. pairs)
MA(1) $X_t = Z_t + \theta Z_{t-1}$ ← CPL ✗ $0<d<\frac{1}{2}$
ARFIMA(0,d,1) $(1-B)^d X_t = Z_t$ B lag X

GLM $\quad l_i(\beta; y_i) = \dfrac{y_i\theta_i - b(\theta_i)}{\phi_i} + c(y_i, \phi_i)$ $\quad y_i \in \mathbb{R}$, $\theta_i \in \Theta$

$$l(\beta, y) = \sum_{i=1}^{n} \Big(\quad \downarrow \quad \Big) \qquad y_i\text{'s ind't}$$

$$\mu_i = \mu_i(\beta) = b'(\theta_i) \qquad g(\mu_i) = x_i^T \beta \qquad x_i \text{ known}$$

MLE

$$\frac{\partial l(\beta; y)}{\partial \beta_j} = \sum_{i=1}^{n} \frac{\partial l_i}{\partial \mu_i}\frac{\partial \mu_i}{\partial \beta_r} = \sum \frac{y_i - \mu_i}{\phi_i V(\mu_i)} \frac{x_{ij}}{g'(\mu_i)} = 0 \qquad j = 1, \dots, p$$

Assume $\phi_i = \phi a_i$, where a_i is known

Can also use model to show $\mathrm{var}(Y_i) = \phi a_i V(\mu_i)$ for some $V(\cdot)$

$$E(Y_i) = \mu_i$$

$EY_i = \mu_i \quad \text{var}\, Y_i = \phi a_i V(\mu_i)$

examples $Y_i \sim N(\mu_i, \sigma^2)$
$a_i = 1 \quad \phi = \sigma^2 \quad V(\mu) = 1$

$Y_i \sim Po(\mu_i) \quad V(\mu_i) = \mu_i$
$a_i = 1 \quad \phi = 1$

$Y_i = R_i / m_i \quad R_i \sim Bin(m_i, p_i)$
$V(p_i) = p_i(1-p_i) \quad \phi = 1 \quad a_i = m_i^{-1}$

$\text{var}\, Y_i = \dfrac{V(p_i)}{m_i}$

$EY_i = p_i$

$Y_i \sim \Gamma(\nu, \mu_i) \quad V(\mu_i) = \mu_i^2$

$$\frac{\partial \ell}{\partial \beta_r} \quad \sum \frac{y_i - \hat{\mu}_i}{a_i V(\hat{\mu}_i)} \frac{x_{ir}}{g'(\hat\mu_i)} = 0 \qquad r = 1, \ldots, p$$

Suppose we instead of using a GLM, we just use it's ml equation as an est'g eq'n:

i.e. assume $EY_i = \mu_i(\beta) \quad g(\mu_i) = x_i^T \beta$
$\text{var}\, Y_i = \phi a_i V(\mu_i)$

& use

$$\sum \frac{Y_i - \mu_i}{a_i V(\mu_i)} \frac{x_{ir}}{g'(\mu_i)} = 0$$

as an est'g eq'n.

the sol'n, $\hat{\beta}$ is free of ϕ

$\text{a.var}\, \hat{\beta} = \phi (X^T W X)^{-1}$

$\hat{\beta} \dot{\sim} N(\beta, \phi(X^T W X)^{-1})$ (Similar to wt'd LS)

$X = \begin{pmatrix} \underset{\sim}{x}_1^T \\ \vdots \\ \underset{\sim}{x}_n^T \end{pmatrix}_{n \times p}$ $W = diag(w_i)$ $w_i = \dfrac{1}{a_i V(\mu_i) g'(\mu_i)^2}$

$U(\hat{\beta}) = 0 \doteq U(\beta) + (\hat{\beta} - \beta)\dfrac{\partial U}{\partial \beta}$

ϕ is an 'over-disp.' parameter

same limit distn as in exp'l family

Fisher inf. in GLM is $\cancel{\phi}(X^T W X)$

$\beta_0 + \beta_1 x_i + \beta_2 x_i^2 + \beta_3 x_i^3$

(orthog polys)

$\dfrac{r_i}{m_i} \sim \cancel{\beta_0}\, \beta_0 + \underbrace{P_1(x_i)}_{\text{linear}} + \underbrace{P_2(x_i)}_{\text{quad.}} + \underbrace{P_3(x_i)}_{\text{cubic}}$

model.matrix(toxo.glm)

est. $\tilde{\phi} = \sum_{i=1}^{n} \frac{(y_i - \mu_i(\hat{\beta}))^2}{(n-p) a_i V(\hat{\mu}_i)}$

Pearson estimate

$\frac{\hat{\beta}_j - \beta_j = 0}{\sqrt{\tilde{\phi} (X^T W^{-1} X)^{-1}_{jj}}}$ e.g. poly(rain, 1) $\hat{\beta} = -.086$

$s.e.\,\hat{\beta} = .467 \times \sqrt{\tilde{\phi}}$

$= .639$

usual var.

Quasi Binom. model deviance red. $\frac{\frac{\overset{33\text{ df}}{74.212} - \overset{30\text{ df}}{62.635}}{3}}{1.94 = \tilde{\phi}}$ $H_0: \beta_1 = \beta_2 = \beta_3 = 0$

F-test

$\sim F_{3,30}$ $p = 0.14$

'dispersion par. est'd'

Binom. $74.212 - 62.635 = 11.8 \sim \chi^2_3$

test of $H: \beta_1 = \beta_2 = \beta_3 = 0$

under Bin. model.

$p \simeq 0.009$

excess variation rel. to Binom.

Very common with propⁿs & $\overline{w}$ counts
that var(Y_i) larger than the model allows

i) new model, e.g. negative binomial, allows for excess variation

ii) use same model, with an 'over dispersion' param.

quasi-likelihood

$$\sum_{i=1}^{n} \frac{y_i-\mu_i}{a_i V(\mu_i)} \frac{\partial \mu_i}{\partial \underset{\sim}{\beta}} = \underset{\sim}{0}$$

$$QL(\beta, \phi) = \sum_{i=1}^{n} \int_{y_i}^{\mu_i} \frac{y_i - u}{\phi\, a_i V(u)}\, du$$

not log of a density all the time

a log-lik. $V(\mu)=\mu^3$

not if $V(\mu)=\mu$

$V(\mu)=\mu(1-\mu)/m$

is if $V(\mu)=\mu^2$

$V(\mu)=1$

If each $\underset{\sim}{y}_i = (y_{i1}, \ldots, y_{in_i})$ e.g. longitudinal data

$E(\underset{\sim}{y}_i) = \underset{\sim}{\mu}_i(\underset{\sim}{\beta})$ $\quad \mathrm{var}(\underset{\sim}{y}_i) = \phi \underset{\sim}{V}(\mu_i, \underset{\sim}{\alpha})$

$n_i \times n_i$ matrix

like a glm, but for possibly dependent components

y_{ij} $\quad j$ indexes time, or group, or..

Score eq^{n}s $\quad \sum_{i=1}^{n} \left(\frac{\partial \underset{\sim}{\mu}_i}{\partial \underset{\sim}{\beta}}\right)^T V(\mu_i, \alpha)^{-1} (\underset{\sim}{y}_i - \underset{\sim}{\mu}_i) = 0$ $\quad \boxed{\sum \frac{(y_i - \mu_i)}{V(\mu_i)} \frac{\partial \mu_i}{\partial \underset{\sim}{\beta}}}$

called GEE $\quad V(\mu_i, \alpha)$? $\quad$ est. α? $\quad$ Diggle, H., Liang, Zeger

$\hat{\beta} \xrightarrow{d} N(\beta, \_\_\_)$