HW #2 due March 10

$$f(y;\mu,\sigma^2) = \frac{1}{\pi\{1+(\frac{y-\mu}{\sigma})^2\}} \cdot \frac{1}{\sigma} \qquad -\infty < y < \infty$$

$\sigma \equiv 1$

$$L(\mu; y) = \prod_{i=1}^{n} \frac{1}{\pi\{1+(y_i-\mu)^2\}}$$

$$\ell(\mu) = \underline{\sum_{i=1}^{n} -\log\{1+(y_i-\mu)^2\} - n\log\pi}$$

$$E\{-\ell''(\mu)\} = i(\mu) \qquad \text{expected Fisher info.}$$

$$= n\, i_1(\mu) \qquad i_1 = E\left[-\log\{1+(Y-\mu)^2\}\right]''$$

$$E(\Sigma X_i) = n\, EX_1 \quad \text{if } X_1, \dots, X_n \text{ iid}$$

$$\ell' \quad \frac{2(y-\mu)}{\{1+(y-\mu)^2\}} \qquad \ell'' = \dots$$

$$\int_{-\infty}^{\infty} \frac{1-(y-\mu)^2}{\{1+(y-\mu)^2\}^2} \cdot \frac{1}{\pi\{1+(y-\mu)^2\}}\, dy \qquad z = y-\mu$$

$$l(\mu,\sigma; y_1,\dots,y_n) = -n\log\sigma - \sum \log\{1+(\tfrac{y_i-\mu}{\sigma})^2\}$$

$$\frac{\partial^2 l}{\partial\mu^2}, \quad \frac{\partial^2 l}{\partial\mu\,\partial\sigma}, \quad \frac{\partial^2 l}{\partial\sigma^2}$$

??

$$E\left(-\frac{\partial^2 l}{\partial\mu\,\partial\sigma}\right) = \quad \cdots \quad = \frac{1}{\sigma^2}\cdot\int\frac{1-z^2}{(1+z^2)^3}\,dz$$

??

m.l.e. $\hat{\mu}$ solves $l'(\hat{\mu})=0$

$$\sum_{i=1}^{n}\frac{(y_i-\hat{\mu})}{\{1+(y_i-\hat{\mu})^2\}} = 0 \qquad \text{defines } \hat{\mu} \quad (\sigma=1)$$

$$\text{a.var}(\hat{\mu}) = \left[E\{-l''(\hat{\mu})\}\right]^{-1} \qquad (\text{likelihood theory})$$

$$\text{a.var}(\widetilde{med}) = ?\ \frac{1}{n4f^2(0)}\ ? \quad \text{for } \tilde{\mu} = \text{med}(y_i)$$

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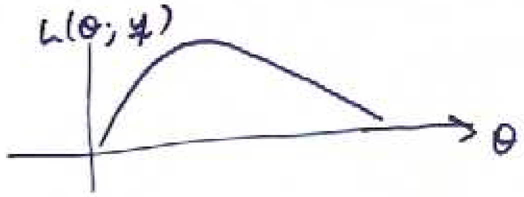
§2.2 General lik. theory, vector param. $\underline{\theta}$

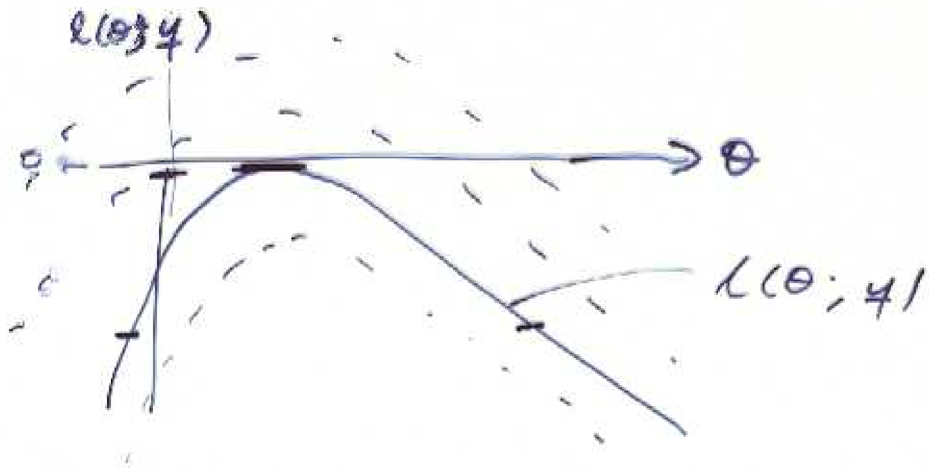
$$\text{Lik } f^n = L(\theta) = \prod_{i=1}^{n} f(y_i;\theta) \propto \prod_{i=1}^{n} f(y_i;\theta) \leftarrow (*)$$

Assume $Y_1,\dots,Y_n$ ind't, id. dist'd, w density $f(y;\theta)$

Function of $\underline{\theta}$



$\ell(\theta)$ = log-likelihood = $\sum_{i=1}^{n} \log f(y_i;\theta) + a(y)$



* ignore constant multiples

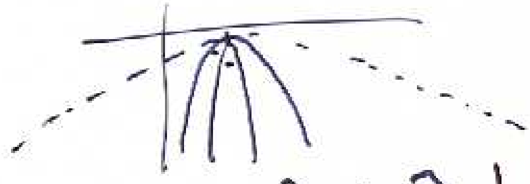
relative values of $\ell(\theta)$ are important but absolute values are not

3 quantities used to summarize this plot

- $\hat\theta$ max. lik. est $\sup_\theta \ell(\theta) = \ell(\hat\theta)$
- $-\ell''(\hat\theta)$ curvature at maximum $(j(\hat\theta))$

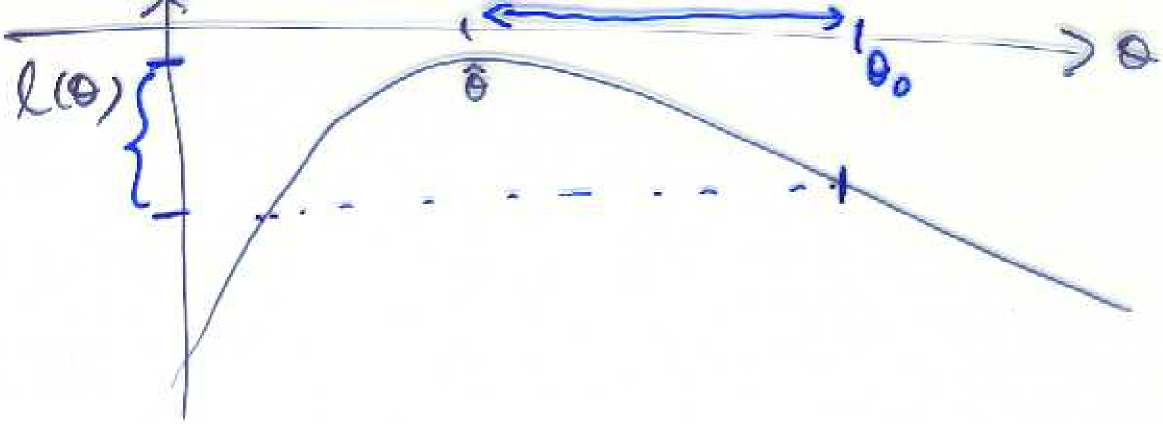
$\hat\theta$ asymptotically$_{n\to\infty}$ $N(\theta, i^{-1}(\theta))$ (*) or ...

where $i(\theta) = E_\theta\, j(\theta)$



- $\ell'(\theta)$ score statistic because we find $\hat\theta$ by

$$\ell'(\hat\theta) = 0$$



$$\hat\theta \;\dot\sim\; N(\theta,\, i^{-1}(\theta)) \qquad (\theta, \sigma^2) \quad \checkmark$$

$$\Rightarrow \quad \hat\theta \pm z_{\alpha/2} \cdot \underset{\substack{\uparrow \\ 1/\hat\sigma}}{j(\hat\theta)^{1/2}} \quad \text{is approx } \begin{matrix}1-\alpha \\ \text{for } \theta\end{matrix} \text{ CI}$$

$$\checkmark \Rightarrow \quad \Pr\left(\frac{|\hat\theta - \theta|}{\{i^{-1}(\theta)\}^{1/2}} \le x \right) \simeq \Pr\left(|N(0,1)| \le x \right)$$

$$\Pr\left(|\hat\theta - \theta| \le z_{\alpha/2}\, i^{-1/2}(\theta) \right) \simeq 1-\alpha \quad \text{if } x = z_{\alpha/2}$$

$$\hat\theta \pm z_{\alpha/2}\, \underbrace{i^{-1/2}(\theta)}_{\downarrow \; \underline{\underline{j^{-1/2}(\hat\theta)}}} \quad \text{is a } 1-\alpha \text{ CI}$$

$\uparrow$ uses a measure of $\hat\theta - \theta_0$ (any fixed θ_0 - value)

Alternative $\quad 2\{\ell(\hat\theta) - \ell(\theta)\} \;\dot\sim\; \chi^2_1$ if θ is 'true' value

How to compute mle's ??

1) scalar $\ell'(\hat{\theta}) = 0 = \ell'(\theta_0) + (\hat{\theta} - \theta_0)\ell''(\theta_0) \cancel{+ \cdots}$

Taylor series

$$\hat{\theta} - \theta_0 \cong -\frac{\ell'(\theta_0)}{\ell''(\theta_0)}$$] Newton-Raphson

$\hat{\theta}^{(0)}$ guess

$$\hat{\theta}^{(t+1)} = \hat{\theta}^{(t)} - \frac{\ell'(\hat{\theta}^{(t)})}{\ell''(\hat{\theta}^{(t)})}$$

$|\hat{\theta}^{(T+1)} - \hat{\theta}^{(T)}| < \underline{\underline{\varepsilon}}$

2) $$\underset{p \times 1}{\underline{\hat{\theta}}^{(t+1)}} = \underset{p \times 1}{\underline{\hat{\theta}}^{(t)}} - \underset{p \times p}{\left[\ell''(\underline{\hat{\theta}}^{(t)})\right]^{-1}} \underset{p \times 1}{\ell'(\underline{\hat{\theta}}^{(t)})}$$

Poisson with no zeroes

$$f(y_i; \lambda) = \frac{\lambda^{y_i} e^{-\lambda}}{y_i! \, (1 - e^{-\lambda})}, \quad y_i = 1, 2, \cdots$$

(he called y_i; n_i)

Vector $\underset{\sim}{\theta}$ (Ho) $\underset{\sim}{\theta}_{p\times 1}$

1) $\hat{\underset{\sim}{\theta}}$: $\sup_{\underset{\sim}{\theta}} \ell(\underset{\sim}{\theta}) = \sup_{\underset{\sim}{\theta}} \ell(\underset{\sim}{\theta}; \underset{\sim}{y})$ ← $y_1, \dots, y_n$

$\hat{\underset{\sim}{\theta}}(y)$

2) $-\ell''(\underset{\sim}{\theta})_{p\times p}$ matrix $\left[-\frac{\partial^2 \ell(\underset{\sim}{\theta})}{\partial\theta_i \partial\theta_j}\right]_{\substack{i=1,\dots p \\ j=1,\dots p}}$

$\{-\ell''(\hat{\theta})\}^{1/2}(\hat{\underset{\sim}{\theta}} - \underset{\sim}{\theta}) \sim N_p(0, I)$

3) $\ell'(\hat{\underset{\sim}{\theta}}) = \underset{\sim}{0}$ ← p eq^{n}s in p unknowns

4) $2\{\ell(\hat{\theta}) - \ell(\theta_0)\} \,\dot{\sim}\, \chi^2_p$ where θ_0 is fixed & $y_1, \dots, y_n \sim f(y; \theta_0)$

Fancier (Hw 2 Qu #1)

$\ell(\hat{\underset{\sim}{\beta}}, \hat{\underset{\sim}{\mu}})$ $\beta_1, \dots, \beta_{10}, \mu_1, \dots, \mu_{10}$ separate gammas

→ $\ell(\hat{\beta}, \hat{\underset{\sim}{\mu}})$ β ; $\mu_1, \dots, \mu_{10}$

4') $2\{\ell(\hat{\underset{\sim}{\beta}}, \hat{\underset{\sim}{\mu}}) - \ell(\hat{\beta}, \hat{\underset{\sim}{\mu}})\} \rightarrow \chi^2_{9=20-11}$

→ $\ell(1, \hat{\underset{\sim}{\mu}})$ $2\{\ell(\hat{\beta}; \hat{\underset{\sim}{\mu}}) - \ell(1; \hat{\underset{\sim}{\mu}})\} \rightarrow \chi^2_1$

$$L(\lambda) = \prod_{i=1}^{n} \frac{\lambda^{y_i} e^{-\lambda}}{y_i! \, (1-e^{-\lambda})}$$

$$\ell(\lambda) = \underline{\sum_{i=1}^{n} y_i} \log\lambda - \underline{\underline{n}}\lambda - \underline{\underline{n}} \log(1-e^{-\lambda}) + c$$

$$= \underset{\text{sum}(n)}{\underline{\underline{y_+}}} \; \underline{\log\lambda} - n\lambda - n\log(1-e^{-\lambda})$$

$$\text{length}(n) \; \{\lambda + \underline{\log(1-e^{-\lambda})}\}$$

nzp.log.likelihood

n <- c(1, 2, 3, 2, 2)

n <- c(1, 2, 3, 2, 2, 5, 7, 1, 6 ...)