Test on Feb 22 Office H 3-4^{30}

Mon Feb 21

GLIModels

response y_i not (usually) normally dist'd

linear predictor $\underset{\sim}{x}_i^T \underset{\sim}{\beta} = \eta_i$ just like linear regression

* link function $\ell(\mu_i) = \eta_i$ where $\mu_i = Ey_i$

$g(\cdot)$ in HO chapter, but $\ell(\cdot)$ in VR

distribution of y_i

$$f(y_i) = \exp\left[\frac{A_i\{y_i\theta_i - \gamma(\theta_i)\}}{\phi} + \tau(y_i, A_i/\phi)\right] \quad (*)$$

A_i known

$\gamma(\cdot)$ is a known $f^{=}$, so is $\tau(\cdot,\cdot)$

ϕ scale parameter

$\theta_i = \theta_i(\mu_i) = \cdots = \theta_i(\underset{\sim}{x}_i^T\underset{\sim}{\beta})$

inverse

$\underline{Ey_i} = \gamma'(\theta_i) = \underline{\mu_i}$

$\underline{\text{var } y_i} = \dfrac{\phi}{A_i}\gamma''(\theta_i)$] ← because of (*)

$= \underline{\dfrac{\phi}{A_i} V(\mu_i)}$

Ex. 1 $\quad y_i \sim N(\mu_i, \sigma^2) \quad \phi = \sigma^2 \quad A_i = 1 \quad \theta_i = \mu_i \quad V(\mu_i) = 1$

$$var(y_i) = \sigma^2 = \frac{\sigma^2}{1} \cdot 1$$

$y_i \sim Poisson(\mu_i) \quad \phi = 1 \quad A_i = 1 \quad \theta_i = \log(\mu_i)$

$$V(\mu_i) = \mu_i$$

$y_i \sim Gamma(\beta, \mu_i) \quad \phi = \frac{1}{\beta} \quad A_i = 1 \quad \theta_i = -\frac{1}{\mu_i}$

$$V(\mu_i) = \mu_i^2$$

ntbc $\quad var(y_i) = \frac{\phi}{A_i} V(\mu_i) = \frac{\mu_i^2}{\beta}$

$$\frac{1}{\Gamma(\beta)} \left(\frac{\beta}{\mu}\right)^{\beta} y^{\beta-1} e^{-y\beta/\mu}$$

$$= \exp -\beta \frac{y}{\mu} + \beta \log(\beta/\mu) + (\beta - 1) \log y_i$$

$y_i \sim Bin(m_i, p_i)$ then let $r_i = y_i / m_i$

$$f(r_i) = \cancel{\exp\left[m_i \{ r_i \log\left(\frac{p_i}{1-p_i}\right) + \log(1-p_i)\right.} \quad m_i \log(1-p_i) + \log m_i$$

$$= \exp\left[m_i \left\{ r_i \log\left(\frac{p_i}{1-p_i}\right) + \log(1-p_i) \right\} + \log \binom{m_i}{m_i r_i}\right]$$

$$\cancel{\left(\frac{p}{1-p}\right)} \quad \cancel{\binom{m}{y} p^y (1-p)^{m-y}}$$

only so $\quad A_i = m_i \quad (\phi = 1)$

$V(\mu_i) = \mu_i(1-\mu_i) \quad \theta_i = \log\left(\frac{\mu_i}{1-\mu_i}\right) \quad \mu_i = E r_i = p_i$

Inference: $y_1, \ldots, y_n$ ind't from $f(y_i)$

inference for β based on likelihood f=

assume ϕ known temporarily

$$L(\beta; y) = \prod_{i=1}^{n} f(y_i; \beta) = \prod_{i=1}^{n} \exp[\qquad]$$

$$= \exp \sum_{i=1}^{n} \left[\frac{A_i\{y_i\theta_i - \gamma(\theta_i)\}}{\phi} + c(y_i, \frac{\phi}{A_i}) \right]$$

$$\ell(\beta; y) = \sum [\qquad] \quad \text{log-lik.}$$

max. lik. estimate solves $\ell'(\hat{\beta}; y) = \underset{\sim}{0}$ $\underline{p \text{ eq'ns}}$

as long as $-\ell''(\hat{\beta})$ is positive semi-def.

for $j = 1, \ldots, p$

$$\frac{\partial \ell}{\partial \beta_j} = \frac{1}{\phi} \sum_{i=1}^{n} A_i \{y_i - \gamma'(\theta_i)\} \left(\frac{\partial \theta_i}{\partial \beta_j}\right)$$

link f= $g(\mu_i) = x_i^T \beta$ $\qquad g'(\mu_i) \frac{\partial \mu_i}{\partial \beta_j} = x_{ij}$ $\qquad \underline{\mu_i = \gamma'(\theta_i)}$

$$\underbrace{g'(\gamma'(\theta_i))}_{g'(\mu_i)} \underbrace{\gamma''(\theta_i)}_{V(\mu_i)} \frac{\partial \theta_i}{\partial \beta_j} = x_{ij}$$

$$= \frac{1}{\phi} \sum_{i=1}^{n} A_i \{y_i - \mu_i\} \frac{x_{ij}}{g'(\mu_i) V(\mu_i)}$$

At mle we have

$$\boxed{\sum_{i=1}^{n} \frac{A_i\,(y_i - \mu_i(\hat\beta))\,x_{ij}}{g'(\mu_i(\hat\beta))\,V(\mu_i(\hat\beta))} = 0} \quad (**) \qquad j=1,\dots,p$$

(Sol'n exists (usually) & determines max. (not min.))

- p. 62 of our HO or eq'n (4.1)
- hidden in Ch. 7 on p. 185 in eq^n (7.4)

text A_i HO $A_i = \frac{1}{a_i}$

$\ell(\cdot)$ $g(\cdot)$ for link.

If $\theta_i = x_i^T\beta \qquad g(\mu_i) = \theta_i \qquad \gamma'(\theta_i) = \mu_i$

$$\Rightarrow g(\gamma'(\theta_i)) = \theta_i$$
$$g'(\gamma'(\theta_i))\,\gamma''(\theta_i) = 1$$
$$g'(\mu_i)\,V(\mu_i) = 1$$

$g(\cdot)$ called "Canonical"

then (**) $\quad \sum A_i\{y_i - \mu_i(\hat\beta)\}x_{ij} = 0 \qquad$ (7.4) VR

$\cancel{\sum A_i y_i = \sum}$

$$\sum_{i=1}^{n} A_i \underline{y_i} x_{ij} = \sum_{i=1}^{n} A_i \underline{\mu_i}(\hat\beta) x_{ij} \qquad j=1,\dots,p$$

"obs'd"   "expected"

$$\sum_{i=1}^{n} A_i \{ \frac{y_i - \mu_i(\hat\beta)}{V(\mu_i(\hat\beta))} \} \frac{x_{ij}}{g'(\mu_i(\hat\beta))} = 0 \qquad j = 1, \dots, p$$

~~Special case: $V(\mu_i) = 1$ $g'(\mu_i) = 1$~~

~~Lets~~

Weighted Least Squares

~~Special case $V(\mu_i) = 1$ $g'(\mu_i) = 1$ $g(\mu_i) = x_i^T\beta$~~

$$\cancel{\sum A_i \{ y_i - \mu_i(\hat\beta) \} x_{ij} = 0}$$

OLS $\quad y = X\beta + \varepsilon \qquad \varepsilon \sim (0, \sigma^2 I)$

$$\hat\beta = (X^T X)^{-1} X^T y$$

Generalize $\quad \varepsilon \sim N(0, \sigma^2 V) \qquad V = \operatorname{diag}(v_1, \dots, v_n)$ known

$$y^* = V^{-1/2} y \qquad \varepsilon^* = V^{-1/2}\varepsilon \qquad X^* = V^{-1/2} X$$

$$V^{-1/2} y = V^{-1} X\beta + V^{-1/2}\varepsilon \qquad y^* = X^*\beta + \varepsilon^*$$

$$\varepsilon^* \sim (0, \sigma^2 I)$$

$$\hat\beta = (X^{*T} X^*)^{-1} X^{*T} y^*$$

$$= (X^T V^{-1} X)^{-1} X^T V^{-1} y \quad \leftarrow \text{weighted LS}$$

$$= (X^T W X)^{-1} X^T W y \qquad W = V^{-1} \text{ weights}$$

You can show sol'n to (7.4) [easy and eg^n]
is a weighted LS sol^=

$$\sum A_i (y_i - \mu_i) x_{ij} = 0 \qquad j = 1, \dots, p$$

Sol^= is $\hat{\beta} = (X^T W X)^{-1} X^T W y$

$$W = \mathrm{diag}(w_i) = \mathrm{diag}(A_i)$$

$\left(\min_\beta \quad \sum A_i (y_i - \mu_i)^2 \right.$ gives (*)

$$\sum \underline{A_i \frac{(y_i - \hat{\mu}_i)}{V(\hat{\mu}_i)} \frac{x_{ij}}{g'(\hat{\mu}_i)}} = 0$$

⇐ iterative WLS scheme

Alg. $\hat{\mu}_i^{(0)} = \mu_i(\hat{\beta}^{(0)})$

step t: $z_i^{(t)} = \hat{\eta}_i^{(t)} + (y_i - \hat{\mu}_i^{(t)}) \cdot g'(\hat{\mu}_i^{(t)})$

Taylor series
$g(\mu_i) = \eta_i$

$$w_i^{(t)} = \frac{A_i}{V(\hat{\mu}_i^{(t)}) \{g'(\hat{\mu}_i^{(t)})\}^2}$$

$$\hat{\beta}^{(t+1)} = (X^T \hat{W}^{(t)} X)^{-1} X^T \hat{W}^{(t)} \hat{z}^{(t)}$$

see top p-64

$E\underline{z_i}^{(t)} = \eta_i^{(t)} \qquad \mathrm{var}\, z_i^{(t)} = \dfrac{V(\mu_i^{(t)}) g'\{\mu_i^{(t)}\}^2}{A_i}$

new problem new y (z) new wts (w) depend on β

glm (y ~ x1 + x2 ... + xp, family = binomial)

y either 2 cols (succ failure)

or it can be a proportion r = succ / total

but if it is a proportion need to input m_i

as 'weight' option ...

cbind (r, m-r) $\leftarrow$ 1st way