STA3000: Pivotal quantities based on profile log-likelihoods

The asymptotic theory outlined in the nuisance parameter notes leads to the following three pivotal quantities, in the case that $\theta = (\psi, \lambda)$ and $\psi \in \mathbb{R}$:

$$\begin{aligned}
r_{\mathrm{p}}(\psi) &= \mathrm{sign}(\hat{\psi} - \psi)[2\{\ell_{\mathrm{p}}(\hat{\psi}) - \ell_{\mathrm{p}}(\psi)\}]^{1/2}, \\
r_e(\psi) &= (\hat{\psi} - \psi) j_{\mathrm{p}}(\hat{\psi})^{1/2}, \\
r_u(\psi) &= \ell_{\mathrm{p}}(\psi) j_{\mathrm{p}}(\hat{\psi})^{-1/2},
\end{aligned}$$

and these are all approximately standard normal pivots, under the model $f(y; \psi, \lambda)$.

[Aside: the score based pivot is not often used, because the normal approximation seems to be poor in many settings. A version of the standardized score statistic that can be useful is the version given in (8) of the nuisance parameter notes:

$$w_u(\psi) = U_\psi(\psi, \hat{\lambda}_\psi)^T \{i^{\psi\psi}(\psi, \hat{\lambda}_\psi)\} U_\psi(\psi, \hat{\lambda}_\psi),$$

because this requires fitting only the model with ψ fixed. For example, if it were of interest to assess whether or not $\psi = 0$, i.e. whether or not the simpler model (without ψ) was just as good as the more complex model, then the score statistic only involves fitting the simpler model. This can be useful in some applications.]

The pivotal quantities r_{p} and r_e are illustrated in Figure 4.7 (lower) in SM (p.130), along with the profile log-likelihood function.

Here is some `R` code that fits a logistic regression to the Challenger shuttle data given in SM as Example 1.3. The model is $y_i \sim Binomial(m_i, p_i)$, where $m_i = 6$, and $\mathrm{logic}(p_i) = \beta_0 + \beta_1 pressure_i + \beta_2 temperature_i$, $i = 1, \ldots, 23$.

```
> library(SMPracticals)
> data(shuttle)
> head(shuttle)
  stability error sign wind   magn vis  use
1     xstab    LX   pp head  Light  no auto
2     xstab    LX   pp head Medium  no auto
3     xstab    LX   pp head Strong  no auto
4     xstab    LX   pp tail  Light  no auto
5     xstab    LX   pp tail Medium  no auto
6     xstab    LX   pp tail Strong  no auto
> ## wrong shuttle data

> data(shuttle, package = "SMPracticals")
> shuttle
   m r temperature pressure
1  6 0          66       50
2  6 1          70       50
3  6 0          69       50
4  6 0          68       50
```

```
...

> attach(shuttle) # simplifies use of names for the next step
> shuttle.glm <- glm (cbind(r,m) ~ temperature + pressure, family = binomial)
> summary(shuttle.glm)

Call:
glm(formula = cbind(r, m) ~ temperature + pressure, family = binomial)

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-0.9783  -0.6438  -0.5428  -0.1144   2.0898

Coefficients:
             Estimate Std. Error z value Pr(>|z|)
(Intercept)  1.696161   3.405617   0.498   0.6185
temperature -0.086153   0.043549  -1.978   0.0479 *
pressure     0.007937   0.007664   1.036   0.3004
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

(Dispersion parameter for binomial family taken to be 1)0

    Null deviance: 21.012  on 22  degrees of freedom
Residual deviance: 14.600  on 20  degrees of freedom
AIC: 34.515

Number of Fisher Scoring iterations: 5
```

A p-value for testing that the coefficient of temperature is zero is given (approximately) by referring the Wald statistic $(\hat{\beta}_1 - 0)j_{\mathrm{p}}^{1/2}(\hat{\beta}_1)$ to a standard normal, and here is 0.048. Similarly the p-value for testing that $\beta_2 = 0$ is approximately 0.300. The likelihood ratio pivot for assessing $\beta_1 = 0$ is obtained by maximizing the log-likelihood function with, and without, that constraint.

```
> glm(cbind(r,m) ~ pressure, family=binomial)

Call:  glm(formula = cbind(r, m) ~ pressure, family = binomial)

Coefficients:
(Intercept)     pressure
  -4.371295     0.009666  -1.9

Degrees of Freedom: 22 Total (i.e. Null);  21 Residual
Null Deviance:     21.01
Residual Deviance: 18.78  AIC: 36.69
```

```
> 2*pnorm(sqrt((18.78-14.60)), lower.tail = F)  # deviance has the "2 times" built in;
                                                 #  the outer 2 is for both tails
[1] 0.0409037
```

With a bit more work, it is possible to get confidence intervals based on the log-likelihood ratio pivot, and for this case the interval for β_1 is $(-0.1787, -0.0035)$, for the Wald pivot it is $(-0.1715, -0.0008)$.