STA 3000F (Fall, 2013)

Homework 2, Complete (October 11; due November 1)

1. Non-uniqueness of ancillary statistics. Suppose that $(Y_1, Z_1), \dots (Y_n, Z_n)$ are independent and identically distributed and follow a bivariate normal distribution with $\mathrm{E}(Y_i) = \mathrm{E}(Z_i) = 0, \mathrm{var}(Y_i) = \mathrm{var}(Z_i) = 1$, and $\mathrm{core}(Y_i, Z_i) = \theta$, $-1 < \theta < 1$. This is an example of a curved exponential family; it can be written in exponential family form, but the two canonical parameters are constrained to one dimension.

   (a) Show that $\sum Z_i^2$ and $\sum Y_i^2$ are each ancillary for θ, but that $T = \sum(Y_i^2 + Z_i^2)$ is not ancillary.

   (b) Derive the first two moments of $T/\sqrt{n}$, and plot the variance of this as a function of θ.

2. *Logistic regression.* Suppose Y_i are independent Bernoulli random variables, with density

$$f(y_i) = p_i^{y_i}(1 - p_i)^{1-y_i}, \quad y = 0, 1,$$

   and that

$$\log \frac{p_i}{1 - p_i} = x_i'\beta,$$

   where x_i and β are vectors of length p.

   (a) Write the joint density of $(y_1, \dots, y_n)$ in exponential family form, and give an expression for the minimal sufficient statistic $S = (S_1, \dots, S_p)$, say.

   (b) Show that the conditional distribution of S_j, given $S_{(-j)}$, depends only on β_j.

3. Suppose that Y_i are independent exponential random variables with $\mathrm{E}(Y_i) = \psi\lambda_i$, and Z_i are independent exponential random variables with $\mathrm{E}(Z_i) = \psi/\lambda_i, \quad i = 1, \dots, n$.

   (a) Find the maximum likelihood estimates of λ_i and ψ.

   (b) Show that $\hat{\psi}$ is not consistent for ψ as $n \to \infty$.

4. *Regression-scale models* Suppose $y = (y_1, \ldots, y_n)^T$ have independent components with density

$$\frac{1}{\sigma} f_0(\frac{y_i - x_i^T\beta}{\sigma}),$$

where $f_0(\cdot)$ is a known density on $\mathbb{R}$. In HW 1 you showed that a is ancillary, where $a_i = (y_i - x_i^T\tilde{\beta})/\tilde{\sigma}$, and the estimators $\tilde{\beta}$ and $\tilde{\sigma}$ are given by

$$\tilde{\beta} = (X^TX)^{-1}X^Ty, \quad \tilde{\sigma}^2 = (y - X\tilde{\beta})^T(y - X\tilde{\beta})/(n-p).$$

(In HW1 we called these $\hat{\beta}$, $\hat{\sigma}$, but I'll use this notation below for the maximum likelihood estimators.)

   (a) Show that under the transformation $y_i \to cy_i + x_i^T b$, where $c > 0$, and $b = (b_1, \ldots, b_p)$ is a vector in $\mathbb{R}^p$, that we have

   $$\tilde{\beta} \to c\tilde{\beta} + b, \quad \tilde{\sigma}^2 \to c^2\tilde{\sigma}^2.$$

   Estimators with this property are called equivariant.

   (b) Show that the associated ancillary statistic $\tilde{a} = (y - X\tilde{\beta})/\tilde{\sigma}$ is invariant under the transformation in (a).

   (c) Show that the maximum likelihood estimators of β and σ are also equivariant, and the associated set of residuals $\hat{a} = (y - X\hat{\beta})/\hat{\sigma}$ is invariant.

   (d) Deduce that the distribution of $\hat{a}$ is free of (β, σ), and thus is also ancillary.

5. *Orthogonal parameters.* In a model $f(y;\theta)$ with $\theta = (\psi, \lambda)$, the component parameters ψ and λ are orthogonal (with respect to expected Fisher information) if $i_{\psi\lambda}(\theta) = 0$.

   (a) Assume y_i follows an exponential distribution with mean $\lambda e^{-\psi x_i}$, where x_i is known. Find conditions on the sequence $\{x_i, i = 1, \ldots, n\}$ in order that λ and ψ are orthogonal with respect to expected Fisher information. Find an expression for the constrained maximum likelihood estimate $\hat{\lambda}_\psi$ and show the effect of parameter orthogonality on the form of the estimate.

(b) Suppose that $y_1, \ldots, y_n$ are independently normally distributed with mean

$$\mathrm{E}(y_i) = \frac{\alpha x_i}{\beta + x_i},$$

where $x_1, \ldots, x_n$ are known constants, and variance σ^2. This is called the Michaelis-Menten model, used in chemical kinetics. Show that (α, σ^2, χ) are mutually orthogonal, where

$$\chi = \sum \frac{\alpha^3 x_i^2}{(\beta + x_i)^3}.$$